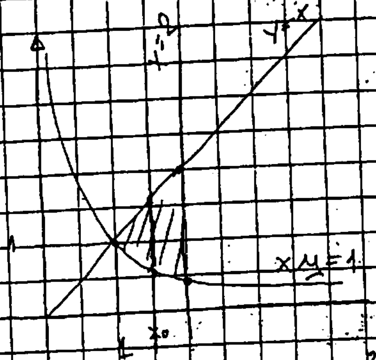
Dvojni integral [DO 6.4B] (N)

1) $I=\iint\limits_D \frac{x^2}{y^2}\,dx\,dy \qquad D:\ y=x,\ xy=1,\ x=2$

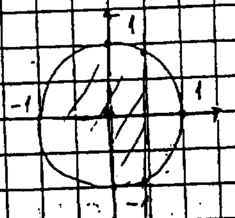


$$D:\begin{cases} 1\le x\le 2 \\ \frac{1}{x}\le y\le x \end{cases}$$

$$I=\int_1^2 dx\int_{1/x}^{x}\frac{x^2}{y^2}\,dy=\int_1^2 dx\left(-\frac{x}{y}\right)\Bigg|_{1/x}^{x}=\int_1^2\left(-\frac{x^2}{x}+\frac{x^2}{\frac{1}{x}}\right)dx=\int_1^2(-x+x^3)\,dx$$

$$=\frac{x^4}{4}-\frac{x^2}{2}\Bigg|_1^2=\frac{16}{4}-\frac{1}{2}\cdot 2+\frac{1}{2}=2+\frac{2-1}{4}=2+\frac{1}{4}=\frac{9}{4}$$

2) $\iint\limits_D y\,dx\,dy \qquad D:\{(x,y)\mid x^2+y^2\le 1\}$



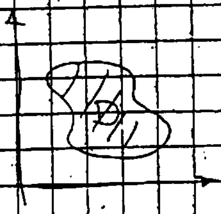
$x^2+y^2=1 \qquad y^2=1-x^2$

$y_1=\sqrt{1-x^2} \qquad y_2=-\sqrt{1-x^2}$

$-1\le x\le 1 \iff -\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$

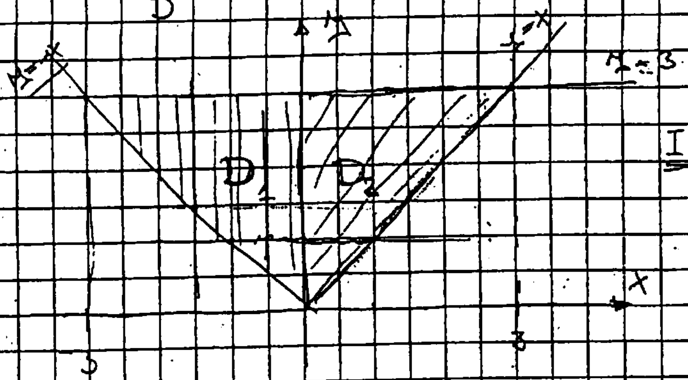
$$I=\int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}y\,dy=\int_{-1}^{1}dx\,\frac{y^2}{2}\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}=\frac{1}{2}\int_{-1}^{1}(1-x^2-1+x^2)\,dx=0$$

НАПОМЕНА:



$$\iint\limits_D dx\,dy=P(D)$$

13) $\iint\limits_D dx\,dy \quad D: y=x,\ y=-x,\ y=3$



I начин: $D_1: -3 \le x \le 0$

$-x \le y \le 3$

$D_2: 0 \le x \le 3$

$x \le y \le 3$

$$I = \iint\limits_D = \iint\limits_{D_1} + \iint\limits_{D_2} = \int_{-3}^{0} dx \int_{-x}^{3} dy + \int_{0}^{3} dx \int_{x}^{3} dy$$

$$= \int_{-3}^{0} y\Big|_{-x}^{3} dx + \int_{0}^{3} y\Big|_{x}^{3} dx$$

$$= \int_{-3}^{0} (3+x)\,dx + \int_{0}^{3} (3-x)\,dx$$

$$= 3x + \frac{x^2}{2}\Big|_{-3}^{0} + 3x - \frac{x^2}{2}\Big|_{0}^{3}$$

$$= +9 - \frac{9}{2} + 9 - \frac{9}{2} = 18 - 9 = 9$$

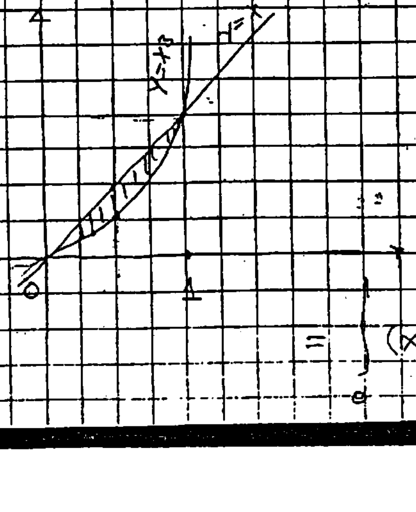
II начин: промена поретка интеграције:

$$D: \begin{cases} 0 \le y \le 3 \\ -y \le x \le y \end{cases} \qquad I = \iint\limits_D dx\,dy = \int_0^3 dy \int_{-y}^{y} dx = \int_0^3 x\Big|_{-y}^{y} dy$$

$$= \int_0^3 y + y\, dy = y^2\Big|_0^3 = 9$$

III начин: $I = P(D) = \frac{6 \cdot 3}{2} = 9.$

4) $\iint\limits_D (x+y)\,dx\,dy \ : \ D: y=x,\ y=x^3,\ x \ge 0$



$0 \le x \le 1$

$x^3 \le y \le x$

$$I = \int_0^1 dx \int_{x^3}^{x} (x+y)\,dy = \int_0^1 dx \left(xy + \frac{y^2}{2}\right)\Big|_{x^3}^{x}$$

$$= \int_0^1 \left(x^2 - x^4 + \frac{x^2}{2} - \frac{x^6}{2}\right) dx = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^3}{6} - \frac{x^7}{14}\Big|_0^1 = \frac{8}{35}$$

I начин: $0 \le y \le 1$, $y \le x \le \sqrt[3]{y}$

$$I = \int_0^1 dy \int_y^{\sqrt[3]{y}} (x+y)\,dx = \int_0^1 dy \left(\frac{x^2}{2} + xy\right)\Big|_y^{y^{1/3}}$$

$$= \int_0^1 \left(\frac{y^{2/3}}{2} - \frac{y^2}{2} + y^{4/3} - y^2\right) dy$$

$$= \frac{3y^{5/3}}{10} - \frac{y^3}{6} + \frac{3y^{7/3}}{7} - \frac{y^3}{3}\Big|_0^1$$

$$= \frac{3}{10} - \frac{1}{6} + \frac{3}{7} - \frac{1}{3} = \frac{21+30}{70} - \frac{1}{2} = \frac{51-35}{70} = \frac{16}{70} = \frac{8}{35}$$

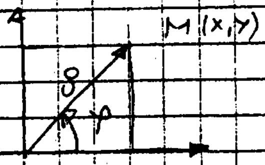
Smena promenljive u dvojnom integralu. 64.

$$I = \iint_D f(x,y)\,dx\,dy \qquad x = x(u,v),\ y = y(u,v)$$

$$J = \begin{vmatrix} x_u' & x_v' \\ y_u' & y_v' \end{vmatrix} = \frac{1}{\begin{vmatrix} u_x' & u_y' \\ v_x' & v_y' \end{vmatrix}} \to I = \iint_D f(u,v)\,|J|\,du\,dv.$$

НАЈЧЕШЋЕ: $x = \rho\cos\varphi$, $0 \le \rho \le +\infty$

$y = \rho\sin\varphi$, $0 \le \varphi \le 2\pi$



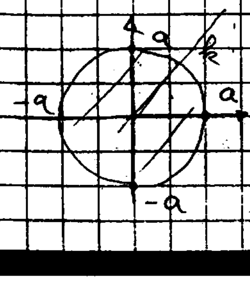
$$J = \begin{vmatrix} x'_\rho & x'_\varphi \\ y'_\rho & y'_\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & +\rho\cos\varphi \end{vmatrix} = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho \quad \boxed{J = \rho}$$

(1-10) Параметризирати следеће области уводећи поларне координате и одредити њихове границе.

(1) $D = \{(x,y) \mid x^2+y^2 \le a^2\}$

$k: x^2+y^2 = a^2$ $\quad \boxed{k: \rho = a}$ – ј-на кружнице у поларним коор.

$$\left.\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right\} \rho^2\cos^2\varphi + \rho^2\sin^2\varphi = a^2 \Leftrightarrow \rho^2 = a^2 \Rightarrow \boxed{\rho = a}$$

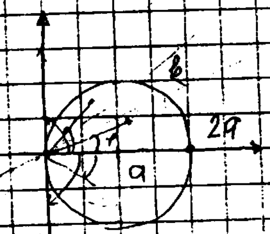


$0 \le \rho \le a$

$0 \le \varphi \le 2\pi$ или $(-\pi \le \varphi \le \pi)$

12. $D = \{(x,y) \mid x^2+y^2 \le 2ax\}$

$k: x^2+y^2 = 2ax \Rightarrow x^2-2ax+y^2=0 : k: (x-a)^2+y^2=a^2$



I НАЧИН $x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$\rho^2\cos^2\varphi - 2a\rho\cos\varphi + \cancel{a^2} + \rho^2\sin^2\varphi = \cancel{a^2}$

$\rho^2 = 2a\rho\cos\varphi \qquad \rho \ne 0$

$\boxed{\rho = 2a\cos\varphi}$

$0 \le \rho \le 2a\cos\varphi$

$-\pi/2 \le \varphi \le \pi/2$

I НАЧИН: $\left.\begin{array}{l} x-a = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right\} \Rightarrow \rho^2 = a^2 \quad \boxed{\rho = a}$

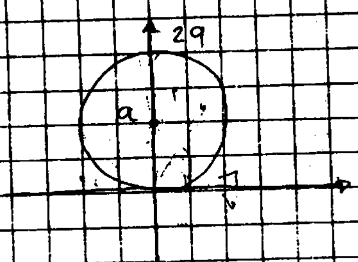
$0 \le \rho \le a$

$0 \le \varphi \le 2\pi$

13. $D = \{(x,y) \mid x^2+y^2 \le 2ay\}$

$x^2+y^2-2ay=0 \Rightarrow x^2+(y-a)^2=a^2$

$k: x^2+(y-a)^2=a^2$

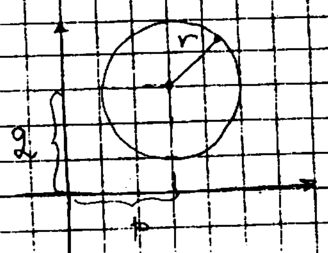


$\left.\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right\} \Rightarrow \rho = 2a\sin\varphi$

$0 \le \rho \le 2a\sin\varphi$

$0 \le \varphi \le \pi$

14. $D = \{(x,y) \mid (x-p)^2+(y-q)^2 = r^2\}$

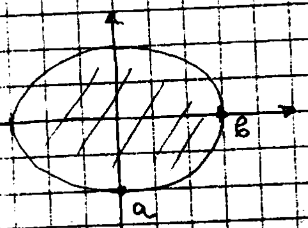


$\left.\begin{array}{l} x-p = \rho\cos\varphi \\ y-q = \rho\sin\varphi \end{array}\right\} \quad \boxed{\rho = r}$

$0 \le \rho \le \rho$

$0 \le \varphi \le 2\pi$

5. $D = \left\{ (x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$



$\left. \begin{array}{l} x = a\rho\cos\varphi \\ y = b\rho\sin\varphi \end{array} \right\} \Rightarrow \frac{a^2\rho^2\cos^2\varphi}{a^2} + \frac{b^2\rho^2\sin^2\varphi}{b^2} = 1$

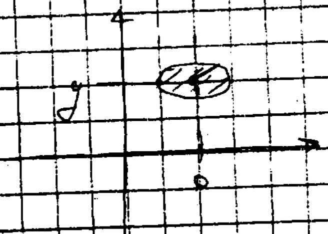
$\boxed{\rho = 1}$

$0 \le \rho \le 1$

$0 \le \varphi \le 2\pi$

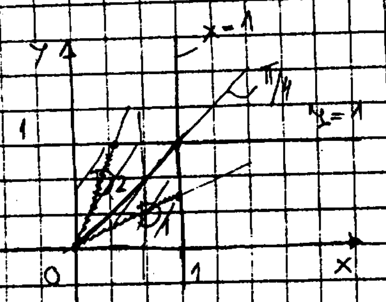
$J = \begin{vmatrix} a\cos\varphi & -a\rho\sin\varphi \\ b\sin\varphi & b\rho\cos\varphi \end{vmatrix} = ab\rho\cos^2\varphi + ab\rho\sin^2\varphi = ab\rho \quad \boxed{J = ab\rho}$

6. $D = \left\{ (x,y) \mid \frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} \le 1 \right\}$



$\left. \begin{array}{l} x - p = a\rho\cos\varphi \\ y - q = b\rho\sin\varphi \end{array} \right\} \Rightarrow \rho = 1 \quad \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \varphi \le 2\pi \end{array}$

7. $D = \{ (x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1 \}$



$D_1: \ 0 \le \varphi \le \pi/4$

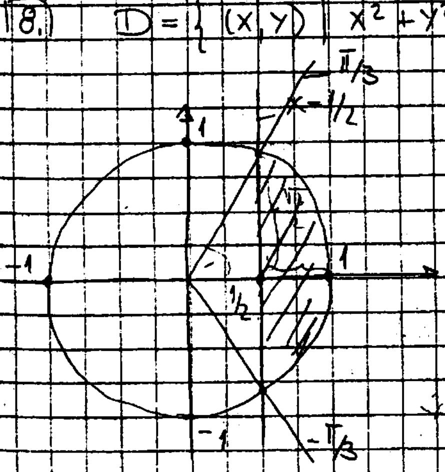
$0 \le \rho \le \frac{1}{\cos\varphi}$

$x = \rho\cos\varphi \quad 1 = \rho\cos\varphi \Rightarrow \rho = \frac{1}{\cos\varphi}$

$D_2: \ \pi/4 \le \varphi \le \pi/2$

$0 \le \rho \le \frac{1}{\sin\varphi} \Leftarrow \quad y = \rho\sin\varphi \quad 1 = \rho\sin\varphi \quad \rho = \frac{1}{\sin\varphi}$

8. $D = \{ (x,y) \mid x^2 + y^2 \le 1,\ x \ge 1/2 \}$



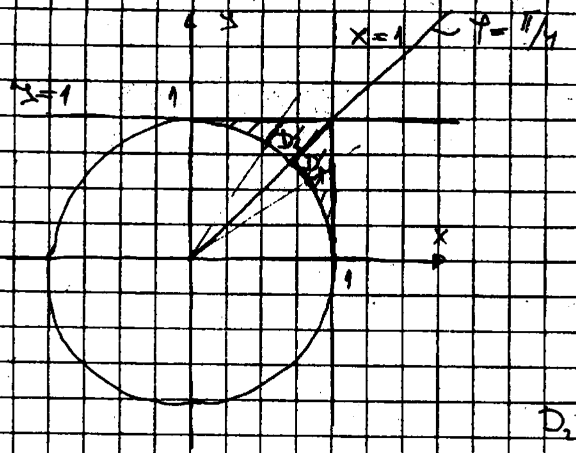
$x = \rho\cos\varphi \qquad x = \frac{1}{2} \Rightarrow \rho\cos\varphi = \frac{1}{2}$

$y = \rho\sin\varphi \qquad \Rightarrow \rho = \frac{1}{2\cos\varphi}$

$\frac{1}{2\cos\varphi} \le \rho \le 1$

$-\frac{\pi}{3} \le \varphi \le \frac{\pi}{3}$

9. $D = \{(x,y) \mid x^2+y^2 \geq 1,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$



$D_1: \quad 0 \leq \varphi \leq \pi/4$

$1 \leq \rho \leq \frac{1}{\cos\varphi}$

$\underline{x=1} \Rightarrow 1 = \rho\cos\varphi \Rightarrow \rho = \frac{1}{\cos\varphi}$

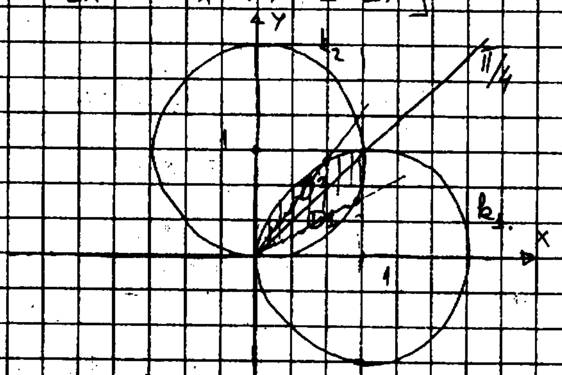
$D_2: \quad \pi/4 \leq \varphi \leq \pi/2$

$1 \leq \rho \leq \frac{1}{\sin\varphi}$

$y=1 \Rightarrow 1 = \rho\sin\varphi$

$\rho = \frac{1}{\sin\varphi}$

10. $D = \{(x,y) \mid x^2+y^2 \leq 2x \ \wedge\ x^2+y^2 \leq 2y\}$



$K_1: \ (x-1)^2+y^2 \leq 1$

$K_2: \ x^2+(y-1)^2 \leq 1$

$D_1: \quad 0 \leq \varphi \leq \pi/4$

$0 \leq \rho \leq 2\sin\varphi$

$2\sin\varphi \Rightarrow \ \rho^2\cos^2\varphi + \rho^2\sin^2\varphi - 2\rho\sin\varphi + \cancel{1} = \cancel{1} \qquad \rho^2 = 2\rho\sin\varphi \Rightarrow \boxed{\rho = 2\sin\varphi}$

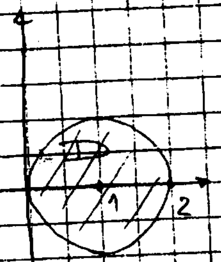
$D_2: \quad \pi/4 \leq \varphi \leq \pi/2$

$0 \leq \rho \leq 2\cos\varphi$

$2\cos\varphi \Rightarrow \ \rho^2\cos^2\varphi - 2\rho\cos\varphi + \cancel{1} + \rho^2\sin^2\varphi = \cancel{1} \qquad \boxed{\rho = 2\cos\varphi}$

① $I = \iint\limits_D \sqrt{x^2+y^2}\,dx\,dy \qquad D = \{(x,y) \mid x^2+y^2 \le 2x\}$

$k_1: x^2+y^2 \le 2x \Rightarrow (x-1)^2+y^2 \le 1$



$\left.\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right\} (\rho\cos\varphi - 1)^2 + \rho^2\sin^2\varphi \Rightarrow$

$\rho^2 - 2\rho\cos\varphi = 1 \Rightarrow \rho = 2\cos\varphi$

$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \qquad |J| = \rho$

$0 \le \rho \le 2\cos\varphi$

$$I = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2\cos\varphi} \sqrt{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi}\,\rho\,d\rho = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2\cos\varphi} \rho^2\,d\rho = \int_{-\pi/2}^{\pi/2} \frac{\rho^3}{3}\Big|_0^{2\cos\varphi} d\varphi$$

$$= \int_{-\pi/2}^{\pi/2} \frac{8\cos^3\varphi}{3} = \frac{16}{3}\int_0^{\pi/2} (1-\sin^2\varphi)\cos\varphi\,d\varphi \quad / \quad \sin\varphi = t$$

$$= \frac{16}{3}\int_0^1 (1-t^2)\,dt = \frac{16}{3}\left[t - \frac{t^3}{3}\right]\Big|_0^1 = \frac{16}{3}\left(1-\frac{1}{3}\right) = \frac{16}{3}\cdot\frac{2}{3} = \frac{32}{9}$$

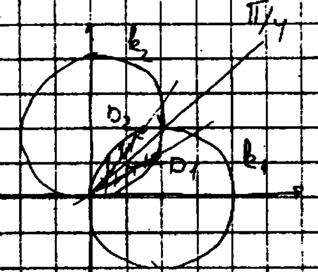
② $I = \iint\limits_D \sqrt{x^2+y^2}\,dx\,dy \qquad D: \{(x,y) \mid x^2+y^2 \le 2x,\ x^2+y^2 \le 2y\}$

$x = \rho\cos\varphi \quad y = \rho\sin\varphi$

$k_1: (x-1)^2+y^2 = 1 \qquad k_1: \rho = 2\cos\varphi$

$k_2: x^2+(y-1)^2 = 1 \qquad k_2: \rho = 2\sin\varphi$

$k_1 \cap k_2 \Rightarrow \cos\varphi = \sin\varphi \Rightarrow \varphi = \pi/4$



$D = D_1 \cup D_2$

$D_1: \quad 0 \le \varphi \le \pi/4 \qquad D_2: \frac{\pi}{4} \le \varphi \le \pi/2$

$0 \le \rho \le 2\sin\varphi \qquad 0 \le \rho \le 2\cos\varphi$

$$I = \iint_{D_1} + \iint_{D_2} = \int_0^{\pi/4} d\varphi \int_0^{2\sin\varphi} \rho^2\,d\rho + \int_{\pi/4}^{\pi/2} d\varphi \int_0^{2\cos\varphi} \rho^2\,d\rho = \int_0^{\pi/4} \frac{\rho^3}{3}\Big|_0^{2\sin\varphi} + \int_{\pi/4}^{\pi/2} d\varphi\,\frac{\rho^3}{3}\Big|_0^{2\cos\varphi}$$

$$= \frac{8}{3}\left[\int_0^{\pi/4} \sin^3\varphi\,d\varphi + \int_{\pi/4}^{\pi/2} \cos^3\varphi\,d\varphi\right] = \triangleright$$

$$I_1 = \int_0^{\pi/4} \sin^3\varphi \, d\varphi = \int_0^{\pi/4} (1-\cos^2\varphi)\sin\varphi \, d\varphi \qquad \begin{array}{l} \cos\varphi = t \\ -\sin\varphi \, d\varphi = dt \end{array}$$

$$= \int_1^{\sqrt{2}/2} (t^2-1)\,dt = \frac{t^3}{3} - t \Big|_1^{\sqrt{2}/2} = \frac{\frac{2\sqrt{2}}{8}}{3} - \frac{1}{3} - \frac{\sqrt{2}}{2} + 1$$

$$= \frac{\sqrt{2}}{12} - \frac{1}{3} - \frac{\sqrt{2}}{2} + 1 = \frac{\sqrt{2} - 4 - 6\sqrt{2} + 12}{12} = \frac{8-5\sqrt{2}}{12}$$

$$I_2 = \int_{\pi/4}^{\pi/2} \cos^3\varphi \, d\varphi = \int_{\pi/4}^{\pi/2} (1-\sin^2\varphi)\cos\varphi \, d\varphi \qquad \begin{array}{l} \sin\varphi = t \\ \cos\varphi \, d\varphi = dt \end{array}$$

$$= \int_{\sqrt{2}/2}^{1} (1-t^2)\,dt = \left(t - \frac{t^3}{3}\right)\Big|_{\frac{\sqrt{2}}{2}}^{1} = 1 - \frac{\sqrt{2}}{2} - \frac{1}{3} + \frac{\sqrt{2}}{12} = \frac{12 - 6\sqrt{2} - 4 + \sqrt{2}}{12} = \frac{8-5\sqrt{2}}{12}$$

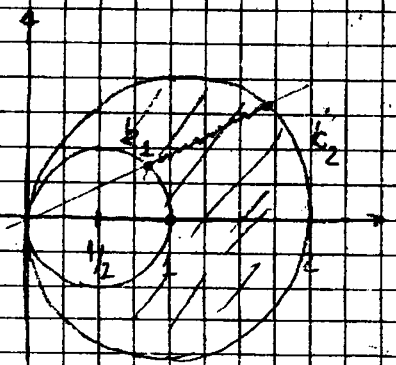
$$I = \frac{8}{3}\left(\frac{8-5\sqrt{2}}{12} + \frac{8-5\sqrt{2}}{12}\right) = \frac{\cancel{8}^{2}}{3} \cdot \frac{(16-10\sqrt{2})}{\cancel{12}\,3} = \frac{2}{9}(16-10\sqrt{2})$$

$$I = \frac{32}{9} - \frac{20\sqrt{2}}{9}$$

3. $\iint_D \sqrt{x^2+y^2}\,dx\,dy \qquad D = \{(x,y) \mid x \le x^2+y^2 \le 2x\}$

$k_1: \; x^2+y^2 = x \Rightarrow (x-1/2)^2 + y^2 = \left(\frac{1}{2}\right)^2 \qquad x = \rho\cos\varphi$

$k_2: \; x^2+y^2 = 2x \Rightarrow (x-1)^2 + y^2 = 1 \qquad y = \rho\sin\varphi$



$k_1: \; \rho^2\cos^2\varphi - \rho\cos\varphi + \cancel{\frac{1}{4}} + \rho^2\sin^2\varphi = \cancel{\frac{1}{4}}$

$\rho^2 - \rho\cos\varphi = 0 \;/\; \boxed{\rho = \cos\varphi}$

$\boxed{k_1: \rho = \cos\varphi}$

$k_2: \; \rho^2\cos^2\varphi - 2\rho\cos\varphi + \cancel{1} + \rho^2\sin^2\varphi = \cancel{1}$

$\boxed{k_2: \rho = 2\cos\varphi}$

$\cos\varphi \le \rho \le 2\cos\varphi$

$-\pi/2 \le \varphi \le \pi/2$

$$I = \int_{-\pi/2}^{\pi/2} d\varphi \int_{\cos\varphi}^{2\cos\varphi} \rho^2 \, d\rho = \int_{-\pi/2}^{\pi/2} \frac{\rho^3}{3}\Big|_{\cos\varphi}^{2\cos\varphi} d\varphi = \int_{-\pi/2}^{\pi/2} \left(\frac{8\cos^3\varphi}{3} - \frac{\cos^3\varphi}{3}\right) d\varphi$$

$$= \frac{7}{3}\int_{-\pi/2}^{\pi/2} \cos^3\varphi \, d\varphi = \frac{14}{3}\int_{0}^{\pi/2} \cos^3\varphi \, d\varphi = \frac{14}{3}\int_{0}^{\pi/2} (1-\sin^2\varphi)\cos\varphi \, d\varphi$$

$\sin\varphi = t$

$\cos\varphi \, d\varphi = dt$

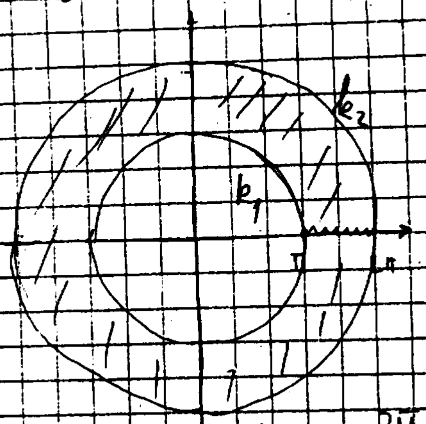
$$\to \frac{14}{3}\int_0^1 (1-t^2)\,dt = \frac{14}{3}\left(t - \frac{t^3}{3}\right)\Big|_0^1 = \frac{14}{3}\left(1-\frac{1}{3}\right) = \frac{14}{3}\cdot\frac{2}{3} = \frac{28}{9}$$

$\boxed{4.}$ $\iint_D \sin\sqrt{x^2+y^2}\, dx\, dy$ $\quad D = \{(x,y) \mid \pi^2 \le x^2+y^2 \le (2\pi)^2\}$

$k_1: x^2+y^2 = \pi^2$ $\quad x = \rho\cos\varphi \quad y = \rho\sin\varphi$

$k_2: x^2+y^2 = (2\pi)^2$ $\quad k_1: \rho = \pi \quad \pi \le \rho \le 2\pi$

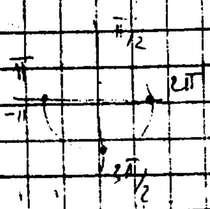
$k_2: \rho = 2\pi \quad 0 \le \varphi \le 2\pi$



$$I = \int_0^{2\pi} d\varphi \int_{\pi}^{2\pi} \rho \sin\rho \, d\rho$$

$\to u = \rho \quad du = d\rho$

$dv = \sin\rho\, d\rho \quad v = -\cos\rho$



$$I = \int_0^{2\pi} d\varphi \left(-\rho\cos\rho\Big|_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos\rho \, d\rho\right)$$

$$I = \int_0^{2\pi} d\varphi \left[(-2\pi\cos 2\pi + \pi\cos\pi) + \cancel{\sin 2\pi}^{\,0} - \cancel{\sin\pi}^{\,0}\right]$$

$$I = \int_0^{2\pi} d\varphi\,(-2\pi+\pi) = -\pi\int_0^{2\pi} d\varphi = -\pi\cdot\varphi\Big|_0^{2\pi} = -2\pi^2$$

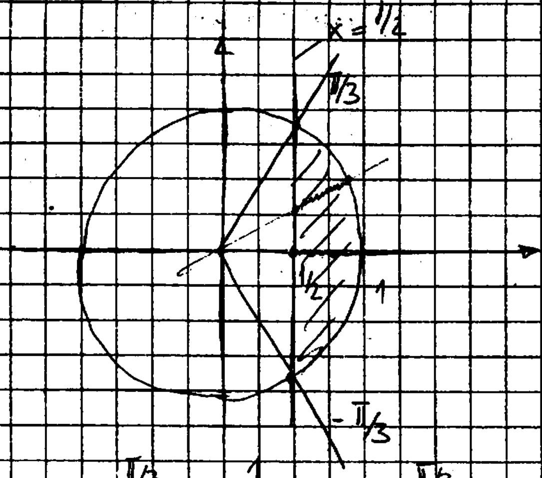
$\boxed{5.}$ $\iint_D (x+y)\,dx\,dy$

a) $D: \{(x,y) \mid x^2+y^2 \le 1, \; x \ge \frac{1}{2}\}$

б) $D: \{(x,y) \mid x^2+y^2 \le 4, \; x^2+y^2 \le 4x\}$

a) $\iint\limits_D (x+y)\,dx\,dy$

$D: \{(x,y) \mid x^2+y^2 \le 1,\ x \ge 1/2\}$



$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

[illegible]

$k: \boxed{\rho = 1}$ $\quad x = \frac{1}{2} = \rho\cos\varphi \ \Big/ \ \rho = \frac{1}{2\cos\varphi}$

$\frac{1}{2\cos\varphi} \le \rho \le 1 \qquad \frac{1}{2\cos\varphi} = 1$

$-\pi/3 \le \varphi \le \pi/3 \qquad \cos\varphi = \frac{1}{2} \quad \varphi = \pm\pi/3$

$$I = \int_{-\pi/3}^{\pi/3} d\varphi \int_{\frac{1}{2\cos\varphi}}^{1} \rho^3\,d\rho = \int_{-\pi/3}^{\pi/3} d\varphi \left(\frac{\rho^4}{4}\right)\Bigg|_{\frac{1}{2\cos\varphi}}^{1} = \int_{-\pi/3}^{\pi/3} \left(\frac{1}{4} - \frac{\frac{1}{16\cos^4\varphi}}{\frac{4}{1}}\right) d\varphi$$

$$= \int_{-\pi/3}^{\pi/3} \left(\frac{1}{4} - \frac{1}{64\cos^4\varphi}\,d\varphi\right)$$

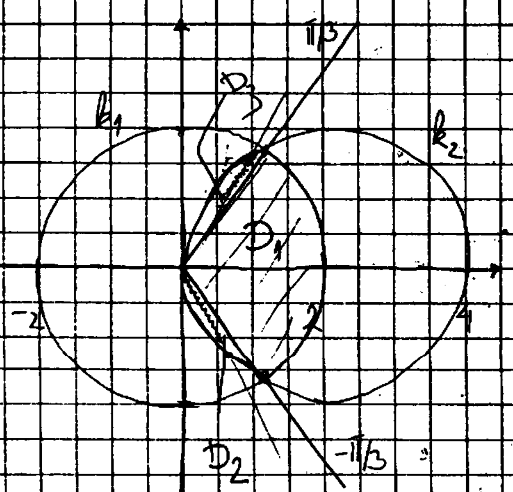
b) $\iint\limits_D \{(x,y) \mid x^2+y^2 \le 4,\ x^2+y^2 \le 4x\}$

$k_1: x^2+y^2=4 \qquad x=\rho\cos\varphi \qquad k_1: \boxed{\rho=2}$

$k_2: (x-2)^2+y^2=4 \qquad y=\rho\sin\varphi \qquad k_2: \rho^2\cos^2\varphi - 4\rho\cos\varphi + \rho^2\sin^2\varphi + \cancel{4} = \cancel{4}$

$k_2: \boxed{\rho = 4\cos\varphi}$

$2 = 4\cos\varphi \qquad \cos\varphi = \frac{1}{2} \qquad \varphi = \pm\pi/3$



$D_1: -\frac{\pi}{3} \le \varphi \le \pi/3$

$0 \le \rho \le 2$

$D_2: -\pi/3 \le \varphi \le -\pi/2$

$0 \le \rho \le 4\cos\varphi$

$D_3: \pi/3 \le \varphi \le \pi/2$

$0 \le \rho \le 4\cos\varphi$

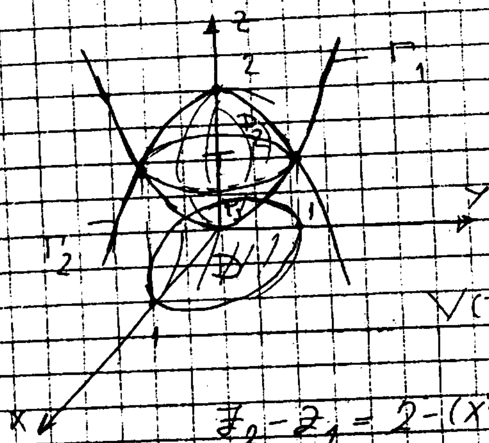
10

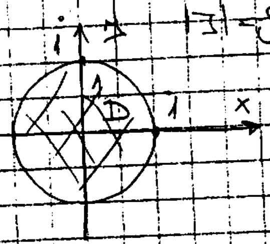
POVRŠINA I ZAPREMINA

1. $V(T)=?\quad P(T)=?$ $\qquad \Gamma_1\cap\Gamma_2 \Rightarrow z=2-(x^2+y^2)$

$\Gamma_1:\ z=x^2+y^2$ $\qquad z=2-z \Rightarrow 2z=2\quad \underline{z=1}$

$\Gamma_2:\ z=2-x^2-y^2=2-(x^2+y^2)$ $\qquad x^2+y^2=1$





$|J|=\rho \quad x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$0\le\rho\le 1,\ 0\le\varphi\le 2\pi$

$$V(T)=\iint_D (z_2-z_1)\,dx\,dy=\int_a^b d\varphi\int_c^d \rho f(x,y)\,d\rho$$

$$z_2-z_1=2-(x^2+y^2)-(x^2+y^2)=2-2(x^2+y^2)=2-2\rho^2$$

$$z_2-z_1=2-2\rho^2$$

$$V(T)=\int_0^{2\pi}d\varphi\int_0^1(2-2\rho^2)\rho\,d\rho=\int_0^{2\pi}d\varphi\int_0^1(2\rho-2\rho^3)\,d\rho$$

$$=\int_0^{2\pi}d\varphi\left(\rho^2-\frac{\rho^4}{2}\right)\Big|_0^1=\int_0^{2\pi}\left(1-\frac{1}{2}\right)d\varphi=\frac{1}{2}\int_0^{2\pi}d\varphi=\frac{1}{2}\varphi\Big|_0^{2\pi}=\pi$$

$$V(T)=\pi$$

$$P=P_1+P_2$$

$$P_1=\iint_D\sqrt{1+p^2+q^2}\,dx\,dy \qquad \Gamma_1:\ z=x^2+y^2 \qquad p=2x \quad q=2y$$

$$P_1=\iint_D\sqrt{1+4x^2+4y^2}\,dx\,dy=\int_0^{2\pi}d\varphi\int_0^1\underbrace{\sqrt{1+4\rho^2}\,\rho\,d\rho}$$

Смена: $1+4\rho^2=t$

$8\rho\,d\rho=dt$

$\Rightarrow \rho\,d\rho=\frac{dt}{8}$

$1+4=5=t$

$1=1$

$$\int_0^{2\pi}d\varphi\int_1^5\frac{t^{1/2}}{8}\,dt=\int_0^{2\pi}d\varphi\,\frac{\sqrt{t^3}}{12}\Big|_1^5=\int_0^{2\pi}\frac{5\sqrt5}{12}-\frac{1}{12}\,d\varphi$$

$$=\frac{5\sqrt5}{12}\cdot 2\pi-\frac{\pi}{6}=\frac{\pi}{6}(5\sqrt5-1)$$

$$\boxed{P_1=\frac{\pi}{6}(5\sqrt5-1)}$$

$P_2 = \iint\limits_D \sqrt{1+p^2+q^2}\,dx\,dy$ $\qquad \Gamma_2 : z = 2 - x^2 - y^2 \quad p = -2x \quad q = -2y$

$P_2 = \iint\limits_D \sqrt{1+4x^2+4y^2}\,dx\,dy = P_1$

$P = P_1 + P_2 = 2P_1 \Rightarrow \boxed{P = \frac{\pi}{3}(5\sqrt{5}-1)}$

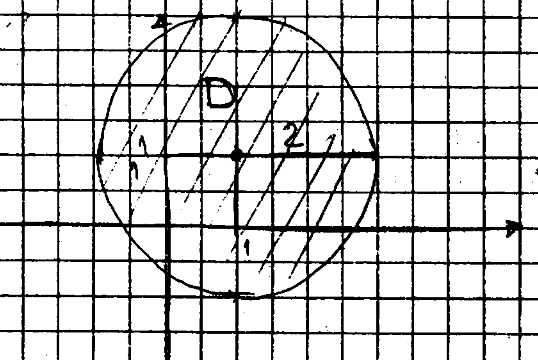
(2) $\Gamma_1 : z = x^2 + y^2$ $\qquad$ скица:

$\Gamma_2 : 2x + 2y - z + 2 = 0$

$\Gamma_1 \cap \Gamma_2 \Rightarrow z = x^2 + y^2$

$2x + 2y - x^2 - y^2 + 2 = 0$

$\Gamma_1 \cap \Gamma_2 : (x-1)^2 + (y-1)^2 = 4$



D: $x - 1 = \rho\cos\varphi$

$y - 1 = \rho\sin\varphi$

$|J| = \rho$

$0 \le \rho \le 2$

$0 \le \varphi \le 2\pi$

$V(T) = \iint\limits_D (2x + 2y + 2 - x^2 - y^2)\,dx\,dy$

$2x + 2y + 2 - x^2 - y^2 = -(x^2 - 2x + y^2 - 2y) + 2$

$= 2 - [(x-1)^2 + (y-1)^2 - 2] = 4 - [(x-1)^2 + (y-1)^2]$

$= 4 - \rho^2$

$V(T) = \int_0^{2\pi} d\varphi \int_0^2 (4-\rho^2)\rho\,d\rho = \int_0^{2\pi} d\varphi \int_4^0 -\frac{t}{2}\,dt = \int_0^{2\pi} \left(-\frac{t^2}{4}\right)\Big|_4^0 d\varphi = \int_0^{2\pi} 4\,d\varphi = 4\varphi\Big|_0^{2\pi}$

$4 - \rho^2 = t \Rightarrow -2\rho\,d\rho = dt$ $\qquad \boxed{V(T) = 8\pi}$

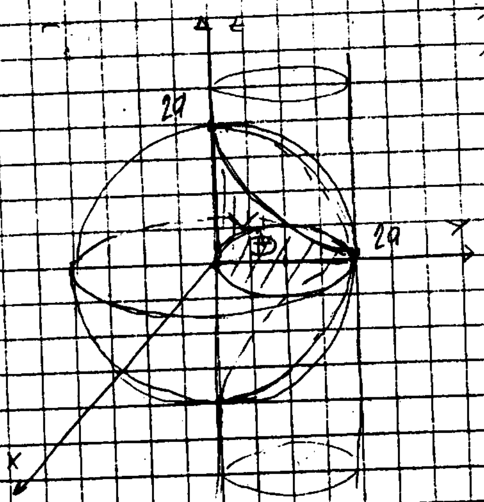
$\rho\,d\rho = -\frac{dt}{2}$

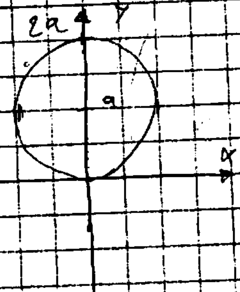
b) $P = \iint_D \sqrt{1+p^2+q^2}\,dx\,dy$ $\quad \Gamma_2: z = 2x+2y+2 \quad p=2,\ q=2$

$$P = \iint_D \sqrt{1+4+4}\,dx\,dy = \iint_D 3\,dx\,dy = 3P(D) = 3r^2\pi = 12\pi$$

3) $\Gamma_1: x^2+y^2+z^2 = 4a^2$

$\Gamma_2: x^2+(y-a)^2 = a^2$





$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$|J| = \rho$

$0 \le \rho \le 2a\sin\varphi$

$0 \le \varphi \le \pi$

$$V_1 = \iint_D (z_{Lopte} - 0)\,dx\,dy$$

$$z_{Lopte} = \sqrt{4a^2-(x^2+y^2)}$$

$$V_1 = \iint_D \sqrt{4a^2-(x^2+y^2)}\,dx\,dy$$

$\cancel{4a^2 - 4a^2\sin^2\varphi = 4a^2\cos^2\varphi}$

$$V_1 = \int_0^{\pi} d\varphi \int_0^{2a\sin\varphi} \sqrt{4a^2-\rho^2}\,\rho\,d\rho =$$

$$\int_0^{2a\sin\varphi} \sqrt{4a^2-\rho^2}\,\rho\,d\rho = \int_0^{2a\sin\varphi} -\frac{t^{1/2}}{2}\,dt = -\frac{\sqrt{t^3}}{3}\Bigg|_0^{2a\sin\varphi} = -\frac{1}{3}\sqrt{(4a^2-\rho^2)^3}\Bigg|_0^{2a\sin\varphi}$$

$4a^2-\rho^2 = t$

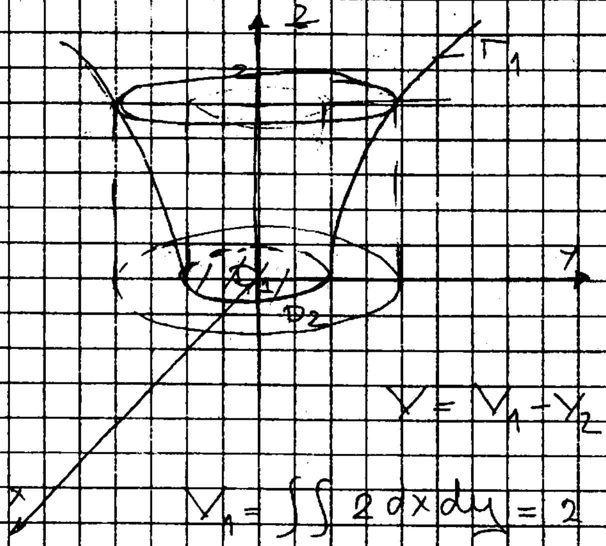
$2\rho\,d\rho = dt \Rightarrow \rho\,d\rho = -\frac{dt}{2}$

$$= -\frac{1}{3}\left[\sqrt{64a^6\cos^6\varphi} - \sqrt{64a^6}\right] = \frac{8a^3}{3} - \frac{8a^3}{3}\cos^3\varphi$$

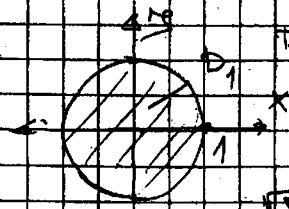
$$V_1 = \int_0^{\pi} \frac{8a^3}{3}\,d\varphi - \cancel{\int_0^{\pi} \frac{8a^3}{3}\cos^3\varphi} = \frac{8a^3}{3}\left(\varphi\Big|_0^{\pi}\right) = \frac{8a^3}{3}\pi$$

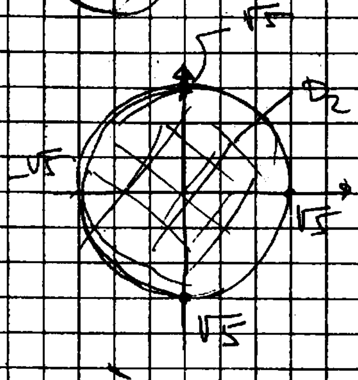
14. Γ_1: $x^2+y^2-z^2=1$

$z=0$, $z=2$



D_1: $x^2+y^2=1$





D_2: $x^2+y^2=5$

$$V = V_1 - V_2$$

$$V_1 = \iint_{D_2} 2\,dx\,dy = 2\iint_{D_2} dx\,dy = 2P(D_2) = 2(\sqrt{5})^2\pi = 10\pi$$

$$V_2 = \iint_{D_2 \setminus D_1} \sqrt{x^2+y^2-1}\,dx\,dy$$

$D_2 \setminus D_1$: $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $0 \le \varphi \le 2\pi$, $1 \le \rho \le \sqrt{5}$

$$V_2 = \int_0^{2\pi} d\varphi \int_1^{\sqrt{5}} \sqrt{\rho^2-1}\,\rho\,d\rho = \frac{16\pi}{3}$$

$\rho^2-1=t$, $2\rho\,d\rho=dt$ $\left\{ \int_0^4 \frac{t^{1/2}}{2}\,dt = \frac{t^{3/2}}{3}\Big|_0^4 = \frac{8}{3} \right.$

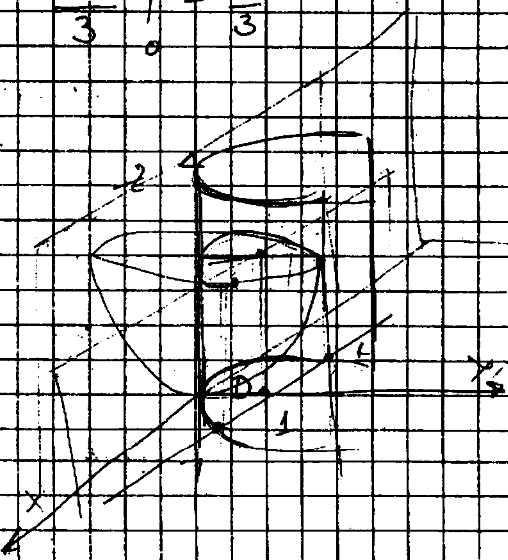
$$V = 10\pi - \frac{16\pi}{3} = \frac{14\pi}{3}$$

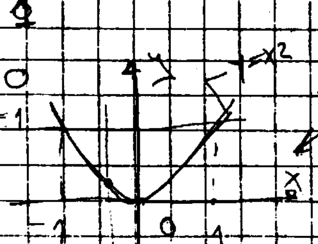
15. Γ_1: $z=x^2+y^2$

Γ_2: $y=x^2$

$y=1$

$z=0$





$M(0,1,0)$

D: $-1 \le x \le 1$, $x^2 \le y \le 1$

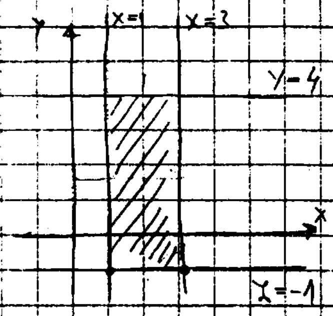
$$V = \int_{-1}^{1} dx \int_{x^2}^{1} (x^2+y^2)\,dy = \frac{88}{105}$$

6.3. Promena poretka integracije u dvojnom integralu.

b)

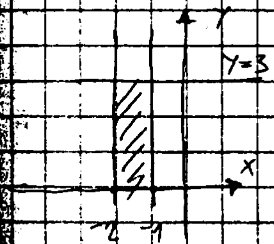
A) Zameniti poredak integracije u sledećem dvojnom integralima.

1. $I=\int_1^3 dx\int_{-1}^4 f(x,y)\,dy$ $\quad 1\le x\le 3,\ -4\le y\le 4$ $\quad\big/\quad$ $-1\le y\le 4,\ 1\le x\le 3$



$$I=\iint_D f(x,y)\,dx\,dy=\int_{-1}^{4}dy\int_1^3 f(x,y)\,dx$$

2. $I=\int_{-2}^{-1}dx\int_0^3 f(x,y)\,dy \Rightarrow$ $\quad -2\le x\le -1,\ 0\le y\le 3$

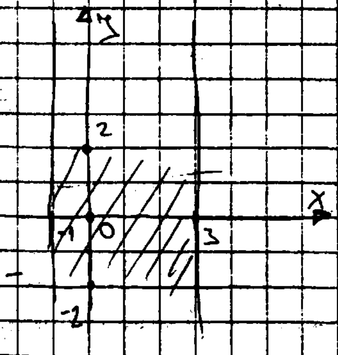


$0\le y\le 3$

$-2\le x\le -1$

$$I=\iint_{D^*} f(x,y)\,dx\,dy=\int_0^3 dy\int_{-2}^{-1} f(x,y)\,dx$$

3. $\int_{-1}^{3}dx\int_{-2}^{2} f(x,y)\,dy \Rightarrow$ $\quad -1\le x\le 3,\ -2\le y\le 2$

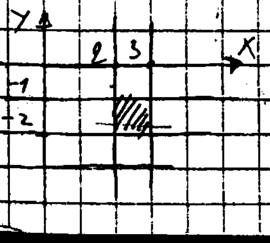


$-2\le y\le 2$

$-1\le x\le 3$

$$I=\iint_{D^*} f(x,y)\,dx\,dy=\int_{-2}^{2}dy\int_{-1}^{3} f(x,y)\,dx$$

4. $\int_2^3 dx\int_{-2}^{-1} f(x,y)\,dy \Rightarrow$ $\quad 2\le x\le 3,\ -2\le y\le -1$



$\Rightarrow$ $-2\le y\le -1$, $2\le x\le 3$

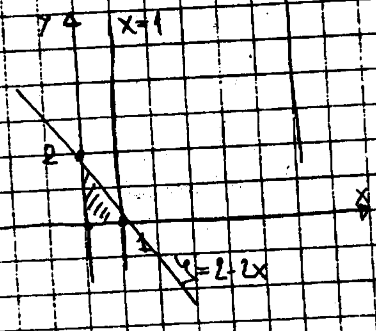
$$I=\iint_{D^*} f(x,y)\,dx\,dy=\int_{-2}^{-1}dy\int_2^3 f(x,y)\,dx$$

5. $I=\int_0^1 dx\int_0^{2-2x} f(x,y)\,dy$

$0\le x\le 1$

$0\le y\le 2-2x$

x	0	1
y	2	0



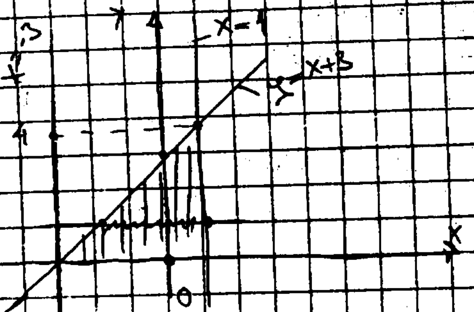
$D^* = \begin{cases} 0\le y\le 2 \\ 0\le x\le 1-\frac{y}{2}\end{cases}$

$I=\iint_{D^*} f(x,y)\,dx\,dy$

$=\int_0^2 dy\int_0^{1-y/2} f(x,y)\,dx$

6. $I=\int_{-3}^{1} dx\int_0^{x+3} f(x,y)\,dy$

$D:\begin{cases} -3\le x\le 1 \\ 0\le y\le x+3\end{cases}$



$D^*:\begin{cases} 0\le y\le 4 \\ y-3\le x\le 1\end{cases}$

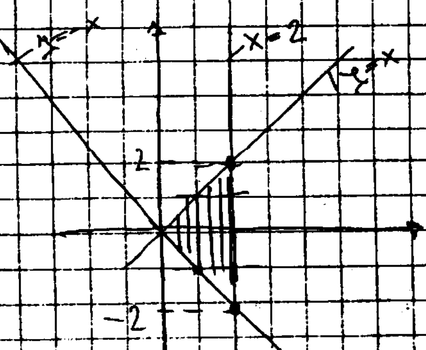
$I=\iint_{D^*} f(x,y)\,dx\,dy$

$=\int_0^4 dy\int_{y-3}^{1} f(x,y)\,dx$

7. $\int_0^2 dx\int_{-x}^{x} f(x,y)\,dy$

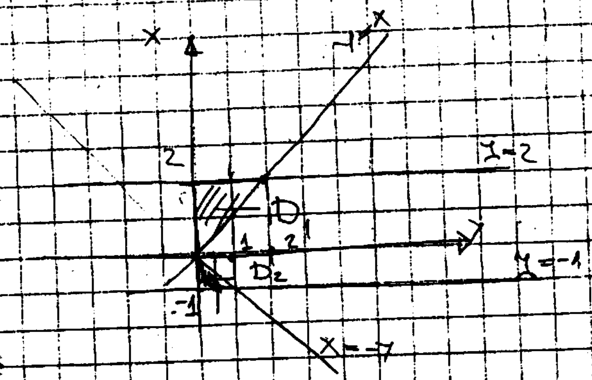
$D:\begin{cases} 0\le x\le 2 \\ -x\le y\le x\end{cases}$

$D_1^*:\begin{cases} -2\le y\le 0 \\ -y\le x\le 2\end{cases}$ $D_2^*:\begin{cases} 0\le y\le 2 \\ y\le x\le 2\end{cases}$



$I=\iint_{D_1^*}+\iint_{D_2^*}=\int_{-2}^{0} dy\int_{-y}^{2} f(x,y)\,dx+\int_0^2 dy\int_y^2 f(x,y)\,dx$

8. $\int_{-1}^{2} dy \int_{0}^{|y|} f(x,y)\,dx$



$D: \begin{cases} -1 \le y \le 2 \\ 0 \le x \le |y| \end{cases}$

$D_1^* = \begin{cases} 0 \le x \le 2 \\ x \le y \le 2 \end{cases}$

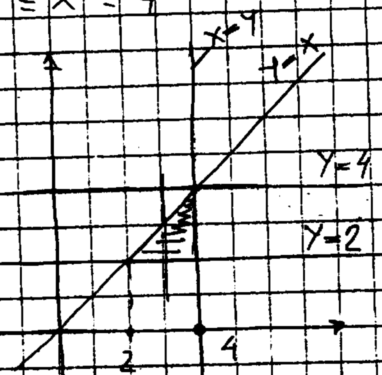
$I = \iint_{D_1^*} + \iint_{D_2^*} = \int_0^2 dx \int_x^2 f(x,y)\,dy + \int_0^1 dx \int_{-1}^{-x} f(x,y)\,dy$

$D_2^* = \begin{cases} 0 \le x \le 1 \\ -1 \le y \le -x \end{cases}$

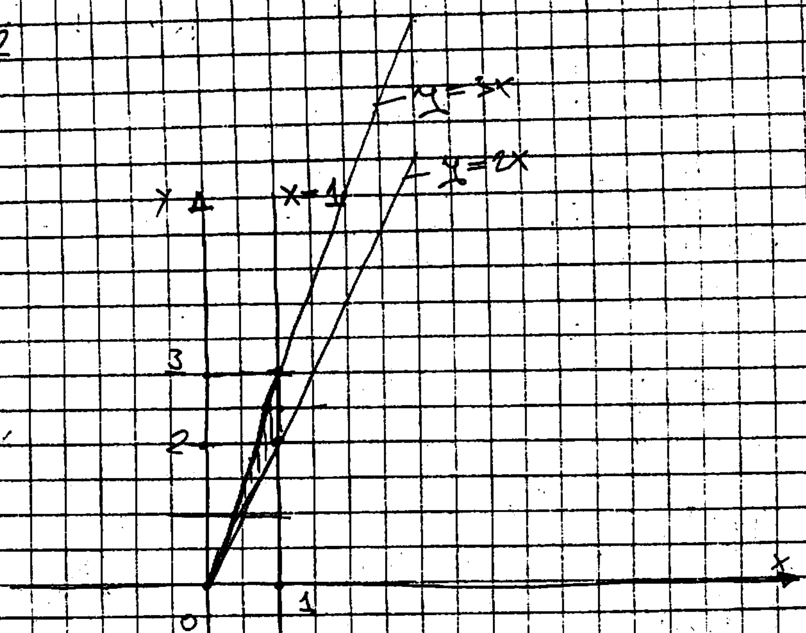
9. $\int_{2}^{4} dy \int_{y}^{4} f(x,y)\,dx$

$D: \begin{cases} 2 \le y \le 4 \\ y \le x \le 4 \end{cases}$

$D^* = \begin{cases} 2 \le x \le 4 \\ 2 \le y \le x \end{cases}$



$I = \iint_{D^*} = \int_2^4 dx \int_2^x f(x,y)\,dy$



10. $\int_{0}^{1} dx \int_{2x}^{3x} f(x,y)\,dy$

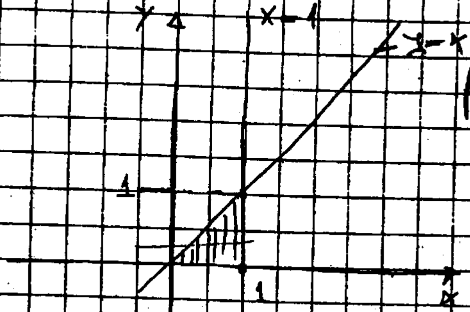
$D: \begin{cases} 0 \le x \le 1 \\ 2x \le y \le 3x \end{cases}$

$D_1^* = \begin{cases} 0 \le y \le 2 \\ \frac{y}{3} \le x \le \frac{y}{2} \end{cases}$

$D_2^* = \begin{cases} 2 \le y \le 3 \\ \frac{y}{3} \le x \le 1 \end{cases}$

$I = \int_0^2 dy \int_{y/3}^{y/2} f(x,y)\,dx + \int_2^3 dy \int_{y/3}^{1} f(x,y)\,dx$

11. $\int_0^1 dx \int_0^x f(x,y)\,dy$

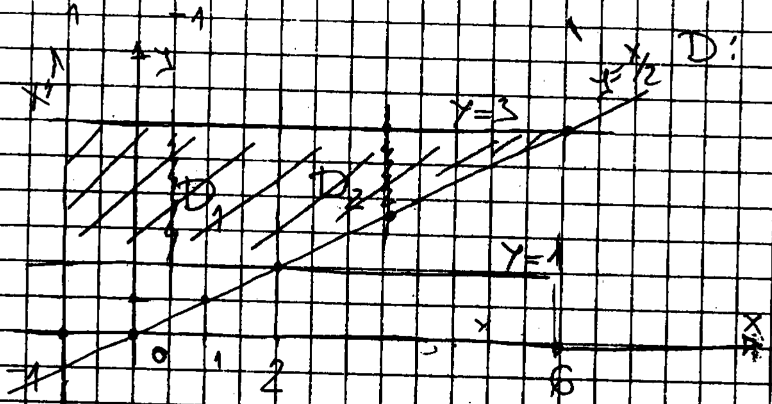


$D : \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$

$D^* : \begin{cases} 0 \le y \le 1 \\ y \le x \le 1 \end{cases}$ $\quad I = \int_0^1 dy \int_y^1 f(x,y)\,dx$

12. $\int_1^3 dy \int_{-1}^{2y} f(x,y)\,dx$

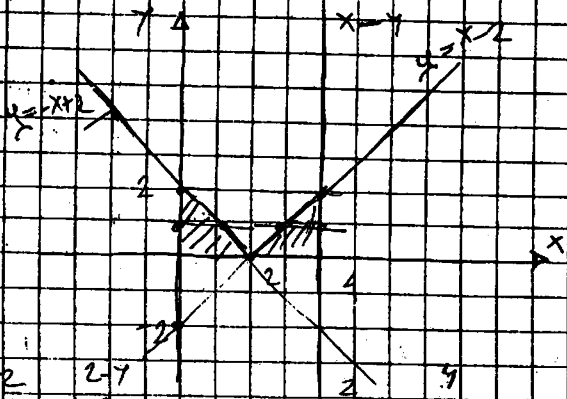
$D : \begin{cases} 1 \le y \le 3 \\ -1 \le x \le 2y \end{cases}$ $\quad y = \frac{x}{2}$



$D_1^* : \begin{cases} -1 \le x \le 2 \\ 1 \le y \le 3 \end{cases}$ $\quad D_2^* : \begin{cases} 2 \le x \le 6 \\ \frac{x}{2} \le y \le 3 \end{cases}$

$I = \int_{-1}^{2} dx \int_1^3 f(x,y)\,dy + \int_2^6 dx \int_{x/2}^{3} f(x,y)\,dy$

13. $\int_0^4 dx \int_0^{|x-2|} f(x,y)\,dy$



$D : \begin{cases} 0 \le x \le 4 \\ 0 \le y \le |x-2| \end{cases}$

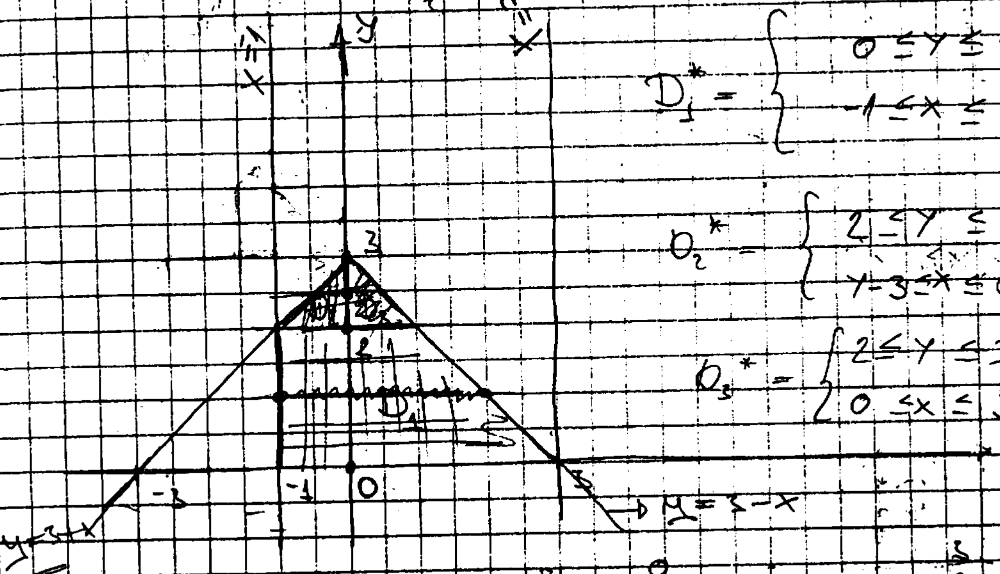
$y = x - 2 \quad y = -x + 2$

$D_1^* = \begin{cases} 0 \le y \le 2 \\ 0 \le x \le 2 - y \end{cases}$

$D_2^* = \begin{cases} 0 \le y \le 2 \\ y + 2 \le x \le 4 \end{cases}$ $\quad I = \int_0^2 dy \int_0^{2-y} f(x,y)\,dx + \int_0^2 dy \int_{y+2}^{4} f(x,y)\,dx$

14. $\displaystyle\int_{-1}^{3} dx \int_{0}^{3-|x|} f(x,y)\,dy$

$$D:\begin{cases} -1 \le x \le 3 \\ 0 \le y \le 3-|x| \end{cases} \qquad y = 3-|x| \begin{cases} x>0 \Rightarrow y = 3-x \\ x<0 \Rightarrow y = 3+x \end{cases}$$



$$D_1^* = \begin{cases} 0 \le y \le 2 \\ -1 \le x \le 3-y \end{cases}$$

$$D_2^* = \begin{cases} 2 \le y \le 3 \\ y-3 \le x \le 0 \end{cases}$$

$$D_3^* = \begin{cases} 2 \le y \le 3 \\ 0 \le x \le 3-y \end{cases}$$

$$I = \int_0^2 dy \int_{-1}^{3-y} f(x,y)\,dx + \int_2^3 dy \int_{y-3}^{0} f(x,y)\,dx + \int_2^3 dy \int_0^{3-y} f(x,y)\,dx$$

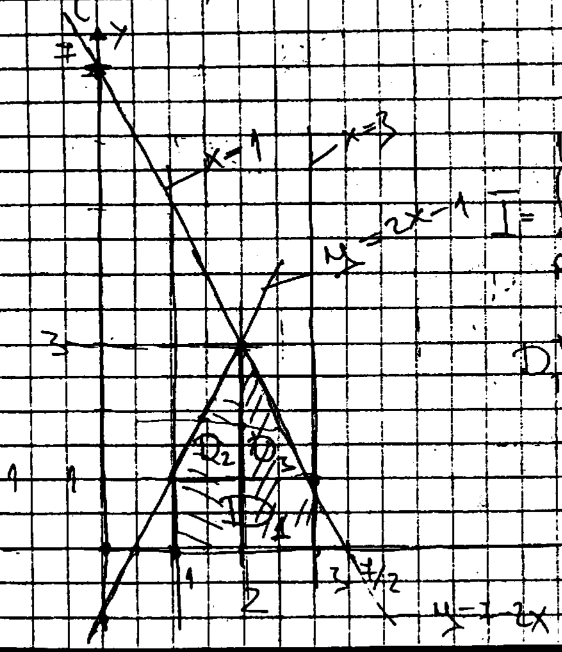
15. $\displaystyle I = \int_1^3 dx \int_0^{3-|2x-4|} f(x,y)\,dy$

$$y = 3-|2x-4| \begin{cases} 2x-4>0,\ x>2 \Rightarrow 3-2x+4 \\ 2x-4<0,\ x<2 \Rightarrow 3+2x-4 \end{cases}$$

$$D:\begin{cases} 1 \le x \le 3 \\ 0 \le y \le 3-|2x-4| \end{cases}$$

$$\Downarrow$$

$$y = \begin{cases} x>2, & y = 7-2x \\ x<2, & y = 2x-1 \end{cases}$$



$$I = \int_0^1 dy \int_1^3 f(x,y)\,dx + \int_1^3 dy \int_{\frac{y+1}{2}}^{2} f(x,y)\,dx + \int_1^3 dy \int_2^{\frac{7-y}{2}} f(x,y)\,dx$$

$$D_1^* = \begin{cases} 0 \le y \le 1 \\ 1 \le x \le 3 \end{cases} \qquad D_2^* \begin{cases} 1 \le y \le 3 \\ \frac{y+1}{2} \le x \le 2 \end{cases}$$

$$D_3^* = \begin{cases} 1 \le y \le 3 \\ 2 \le x \le \frac{7-y}{2} \end{cases}$$

16. $\int_{-5}^{0} dy \int_{0}^{|y+2|} f(x,y)\,dx$

$x = |y+2| = \begin{cases} x = y+2, & y+2>0 \quad y>-2 \\ x = -y-2, & y+2<0 \quad y<-2 \end{cases}$

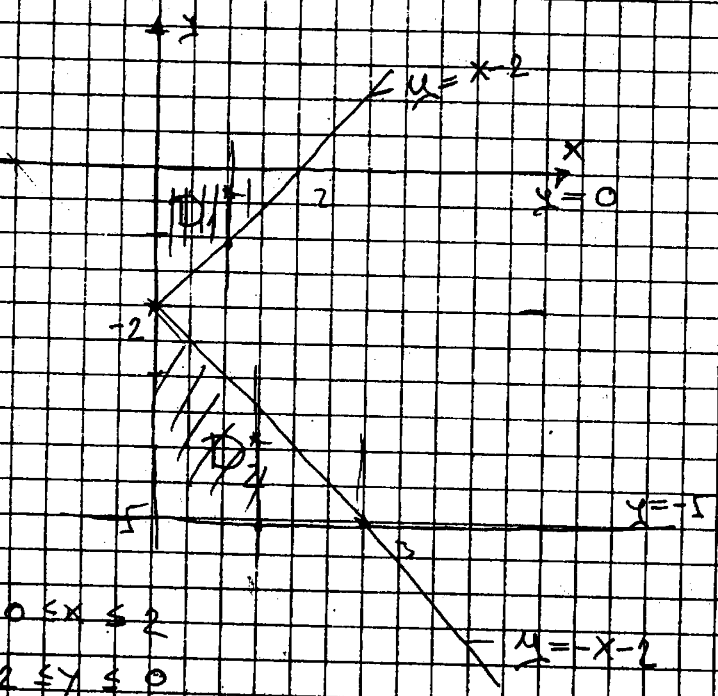
$D = \begin{cases} -5 \le y \le 0 \\ 0 \le x \le |y+2| \end{cases}$

1° $x-2=y$

x	0	2
y	-2	0

2° $y=-x-2$

x	0	-2
y	-2	0

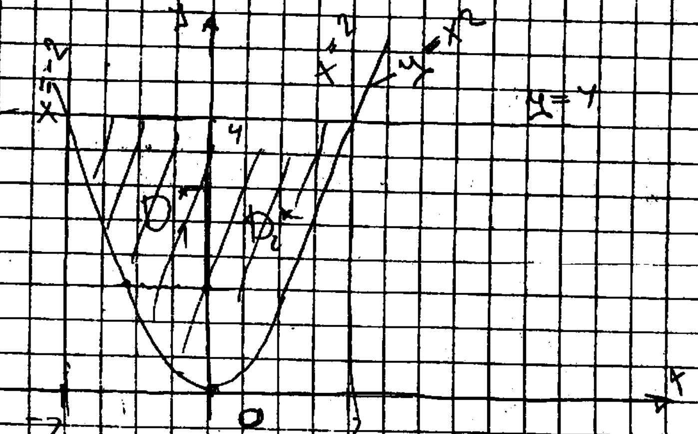


$D_1^* = \begin{cases} 0 \le x \le 2 \\ x-2 \le y \le 0 \end{cases}$

$D_2^* = \begin{cases} 0 \le x \le 3 \\ -5 \le y \le -x-2 \end{cases}$

$I = \int_0^2 dx \int_{x-2}^{0} f(x,y)\,dy + \int_0^3 dx \int_{-5}^{-x-2} f(x,y)\,dy$

17. $\int_{-2}^{2} dx \int_{x^2}^{4} f(x,y)\,dy$



$D: \begin{cases} -2 \le x \le 2 \\ x^2 \le y \le 4 \end{cases}$

$D_1^* = \begin{cases} 0 \le y \le 4 \\ -\sqrt{y} \le x \le 0 \end{cases}$

$D_2^* = \begin{cases} 0 \le y \le 4 \\ 0 \le x \le \sqrt{y} \end{cases}$

18. $\int_0^4 dx \int_{3x^2}^{12x} f(x,y)\,dy$

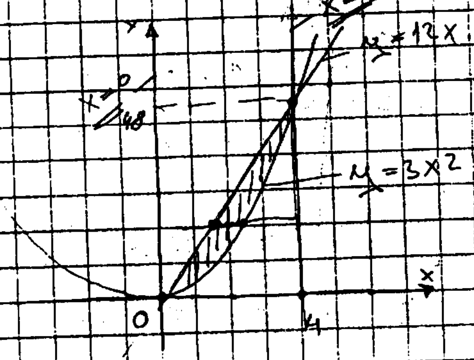
1. $x=0$ 3. $y=3x^2$
2. $x=4$ 4. $y=12x$

$3x^2 = 12x$ ... $3x^2-12x=0$ $3x(x-4)$

$x_1=0,\ x_2=4$

$$D = \begin{cases} 0 \le x \le 4 \\ 3x^2 \le y \le 12x \end{cases}$$

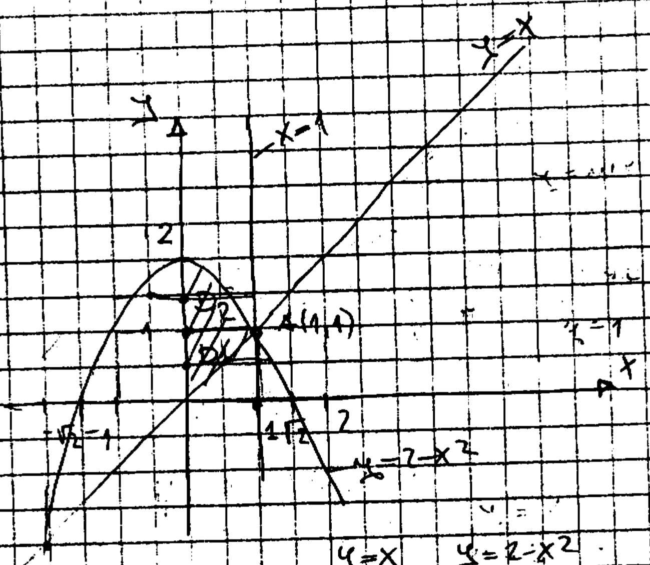
$$D^* = \begin{cases} 0 \le y \le 48 \\ \frac{y}{12} \le x \le \sqrt{\frac{y}{3}} \end{cases}$$



$$I = \int_0^{48} dy \int_{\frac{y}{12}}^{\frac{\sqrt{y}}{\sqrt{3}}} f(x,y)\,dx$$

19. $\int_0^1 dx \int_x^{2-x^2} f(x,y)\,dy$

$$D: \begin{cases} 0 \le x \le 1 \\ x \le y \le 2-x^2 \end{cases}$$



$y=x$ $y=2-x^2$

$2-x^2=x$ $0=x^2+x-2$

$x_{1/2} = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm 3}{2}$

$x_1=-2$ $x_2=1$

$$D_1^* = \begin{cases} 0 \le y \le 1 \\ 0 \le x \le y \end{cases}$$

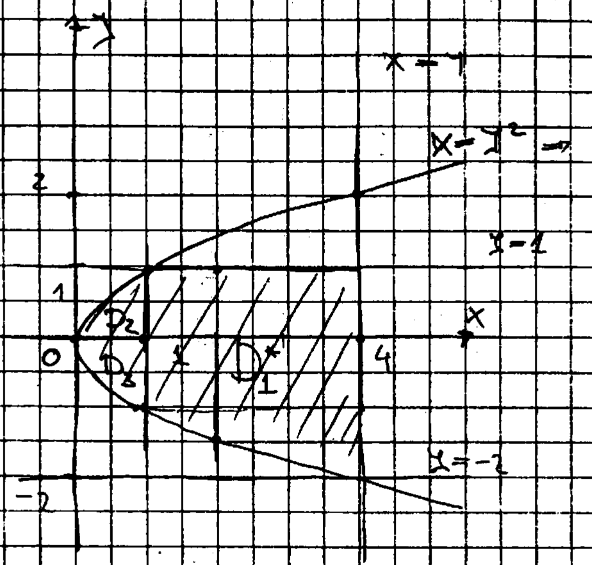
$$D_2^* = \begin{cases} 1 \le y \le 2 \\ 0 \le x \le \sqrt{2-y} \end{cases}$$

$$I = \int_0^1 dy \int_0^y f(x,y)\,dx + \int_1^2 dy \int_0^{\sqrt{2-y}} f(x,y)\,dx$$

20. $\int_{-2}^{1} dy \int_{y^2}^{4} f(x,y)\,dx$ D: $\begin{cases} -2 \le y \le 1 \\ y^2 \le x \le 4 \end{cases}$

$x = y^2$ $x = 4$

$4 = y^2$ $y = \pm 2$



$D_1^* = \begin{cases} 1 \le x \le 4 \\ -\sqrt{x} \le y \le 1 \end{cases}$

$D_2^* = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le \sqrt{x} \end{cases}$

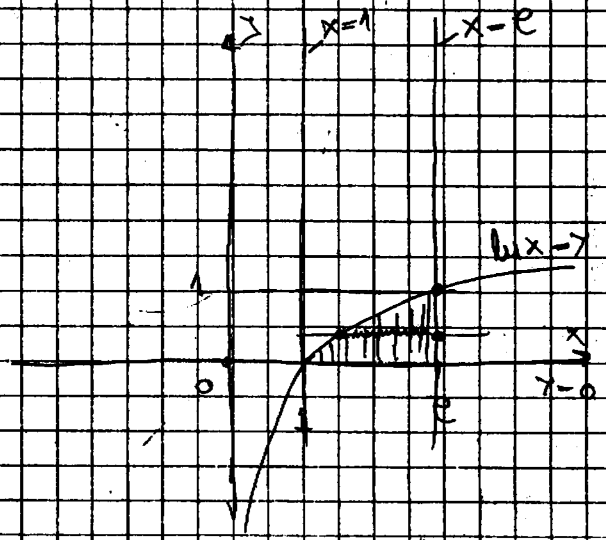
$D_3^* = \begin{cases} 0 \le x \le 1 \\ -\sqrt{x} \le y \le 0 \end{cases}$

21. $I = \int_{1}^{e} dx \int_{0}^{\ln x} f(x,y)$

$D = \begin{cases} 1 \le x \le e \\ 0 \le y \le \ln x \end{cases}$

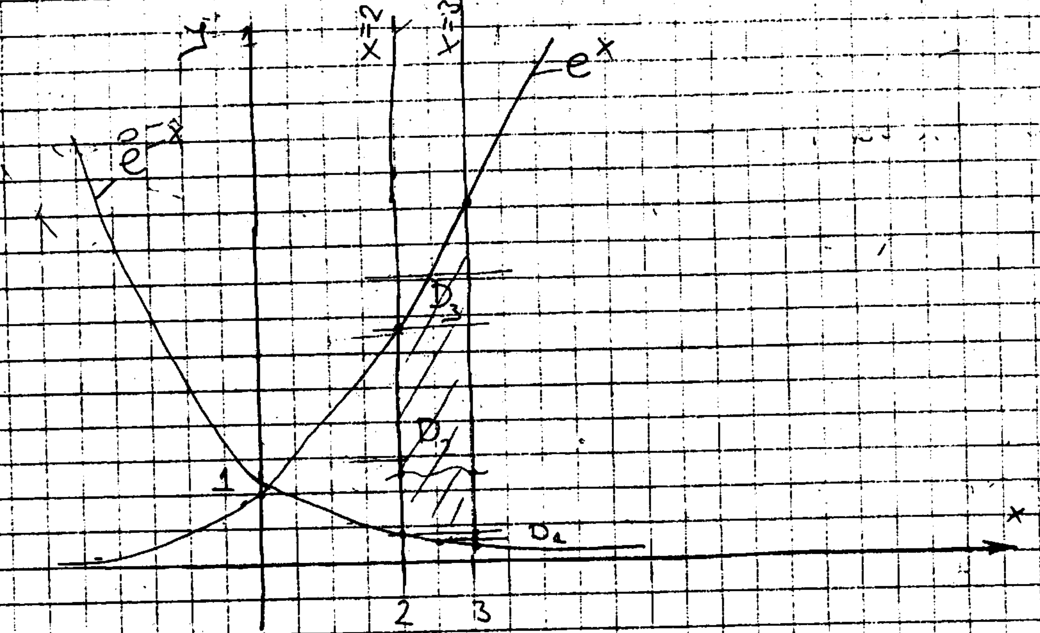
$D^* = \begin{cases} 0 \le y \le 1 \\ e^y \le x \le e \end{cases}$

$I = \int_{0}^{1} dy \int_{e^y}^{e} f(x,y)\,dx$



22. $I = \int_{2}^{3} dx \int_{e^{-x}}^{e^{x}} f(x,y)\,dy$

D: $\begin{cases} 2 \le x \le 3 \\ e^{-x} \le y \le e^{x} \end{cases}$

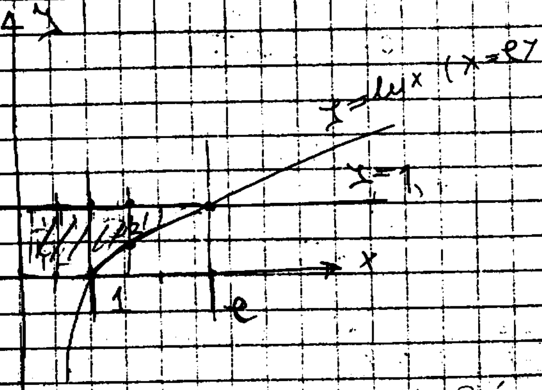


$D_1^* = \begin{cases} e^{-3} \le y \le e^{-2} \\ \ln\frac{1}{y} \le x \le 3 \end{cases}$ $\quad D_2^* = \begin{cases} e^{-2} \le y \le e^{2} \\ 2 \le x \le 3 \end{cases}$

$D_3 = \begin{cases} e^{2} \le y \le e^{3} \\ \ln y \le x \le 3 \end{cases}$

$$I = \int_{e^{-3}}^{e^{-2}} dy \int_{\ln 1/y}^{3} f(x,y)\,dx + \int_{e^{-2}}^{e^{2}} dy \int_{2}^{3} f(x,y)\,dx + \int_{e^{2}}^{e^{3}} dy \int_{\ln y}^{3} f(x,y)\,dx$$

23. $I = \int_0^1 dy \int_0^{e^y} f(x,y)\,dx$

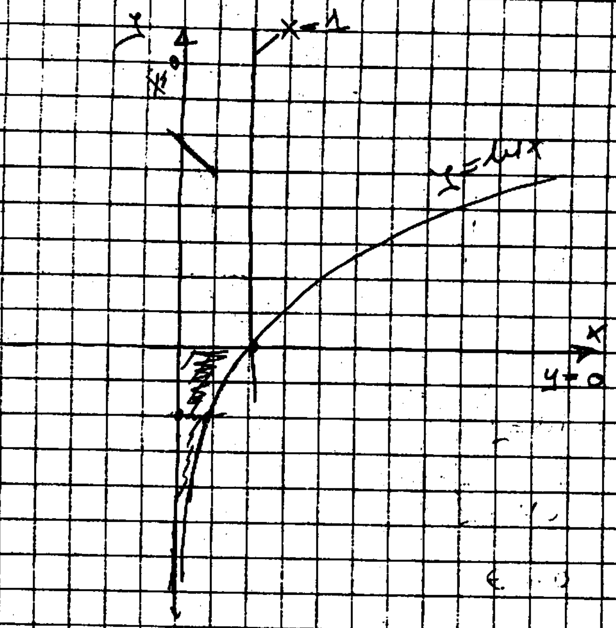


$\begin{cases} 0 \le y \le 1 \\ 0 \le x \le e^{y} \end{cases}$

$D_1^* = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \end{cases}$ $\quad D_2 = \begin{cases} 1 \le x \le e \\ \ln x \le y \le 1 \end{cases}$

$$I = \int_0^1 dx \int_0^1 f(x,y)\,dy + \int_1^e dx \int_{\ln x}^{1} f(x,y)\,dy$$

24. $\int_0^1 dx \int_{\ln x}^{0} f(x,y)\,dy$

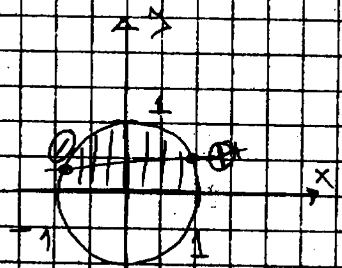


$$D = \begin{cases} 0 \le x \le 1 \\ \ln x \le y \le 0 \end{cases}$$

$$D^* = \Big\{$$

25. $\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} f(x,y)\,dx$

$$D: \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{cases}$$



$y = 0$

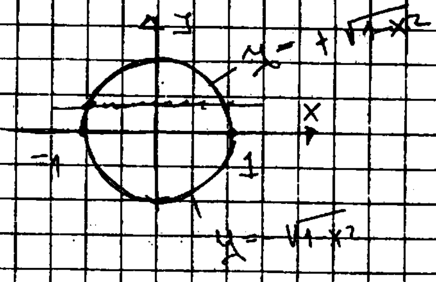
$y^2 = 1 - x^2$

$y^2 + x^2 = 1$

$$D^* = \begin{cases} 0 \le y \le 1 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{cases}$$

$$I = \int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$$

26. $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy$

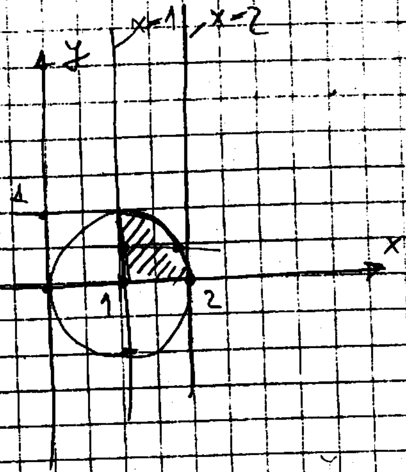


$$D \begin{cases} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{cases}$$

$$D^* = \begin{cases} -1 \le y \le +1 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{cases}$$

$$I = \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$$

27. $\int_1^2 dx \int_0^{\sqrt{2x-x^2}} f(x,y)\,dy$



$D: \begin{cases} 1 \le x \le 2 \\ 0 \le y \le \sqrt{2x-x^2} \end{cases}$

$y^2 = 2x - x^2$

$x^2 - 2x + y^2 = 0$

$(x-1)^2 + y^2 = 1$

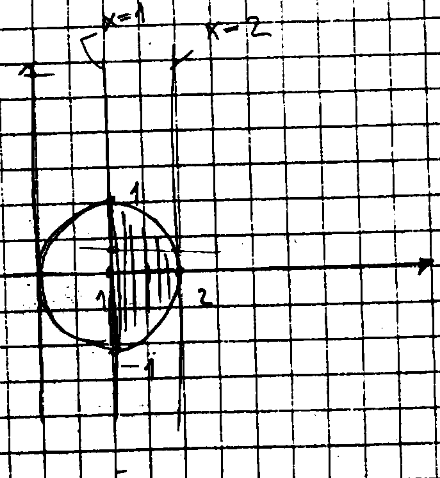
$D^* = \begin{cases} 0 \le y \le 1 \\ 1 \le x \le 1+\sqrt{1-y^2} \end{cases}$

$(x-1)^2 = 1-y^2$

$x-1 = \sqrt{1-y^2}$

$x = 1+\sqrt{1-y^2}$

$$I = \int_0^1 dy \int_1^{1+\sqrt{1-y^2}} f(x,y)\,dx$$



28. $\int_1^2 dx \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} f(x,y)\,dy$

$D = \begin{cases} 1 \le x \le 2 \\ \sqrt{2x-x^2} \le y \le -\sqrt{2x-x^2} \end{cases}$

$D^* = \begin{cases} -1 \le y \le 1 \\ 1 \le x \le 1+\sqrt{1-y^2} \end{cases}$

$$I = \int_{-1}^1 dy \int_1^{1+\sqrt{1-y^2}} f(x,y)\,dx$$

29. $I = \int_{-7}^1 dy \int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x,y)\,dx$

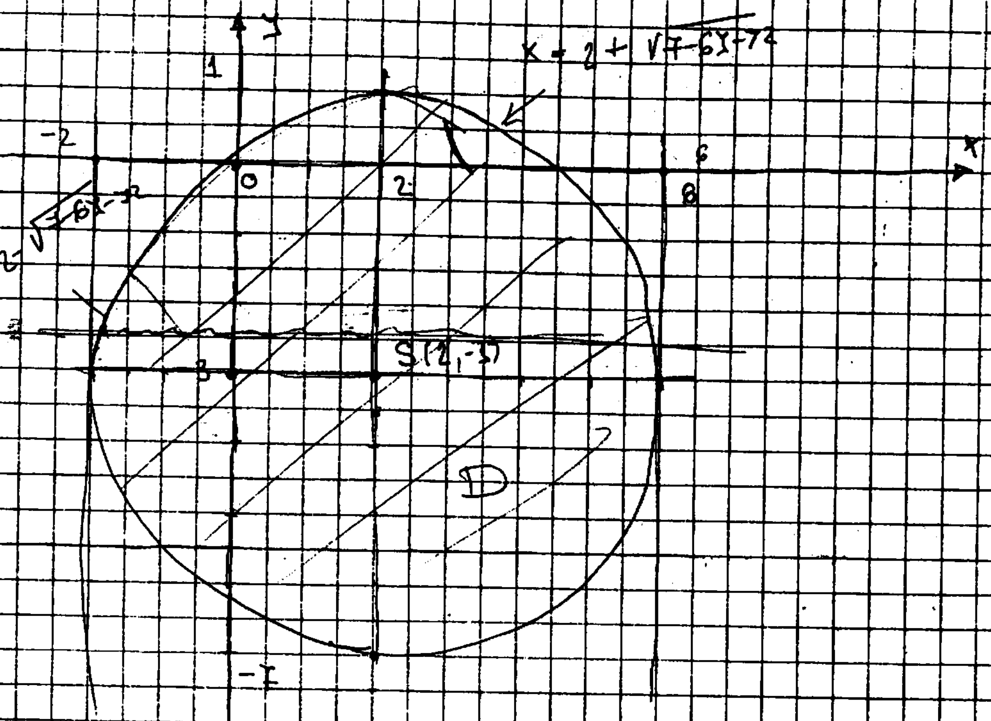
$x = 2 - \sqrt{7-6y-y^2}$

$\begin{cases} -7 \le y \le 1 \\ 2+\sqrt{7-6y-y^2} \le x \le 2-\sqrt{7-6y-y^2} \end{cases}$

$(x-2)^2 = 7-6y-y^2$

$(x-2)^2 + y^2 + 6y - 7 = 0$

$(x-2)^2 + (y+3)^2 = 16$



$(x-2)^2 + (y+3)^2 = 16$

$y+3 = \pm\sqrt{16-(x-2)^2}$

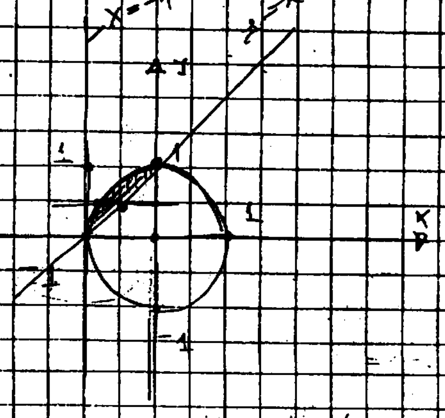
$y = -3 \pm \sqrt{16 - x^2 + 4x - 4}$

$y = -3 \pm \sqrt{12 - x^2 + 4x}$

$$D^* = \begin{cases} -2 \le x \le 6 \\ -3-\sqrt{12-x^2+4x} \le y \le -3+\sqrt{12-x^2+4x} \end{cases}$$

$$I = \int_{-2}^{6} dx \int_{-3-\sqrt{12-x^2+4x}}^{-3+\sqrt{12-x^2+4x}} f(x,y)\,dy$$

30. $$\int_{-1}^{0} dx \int_{x+1}^{\sqrt{1-x^2}} f(x,y)\,dy$$



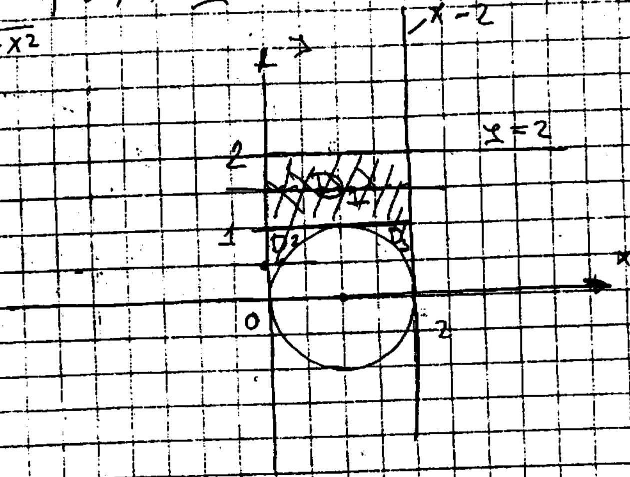
$$D = \begin{cases} -1 \le x \le 0 \\ x+1 \le y \le \sqrt{1-x^2} \end{cases}$$

$$D^* = \begin{cases} 0 \le y \le 1 \\ -\sqrt{1-y^2} \le x \le y-1 \end{cases}$$

$$I = \int_{0}^{1} dy \int_{-\sqrt{1-y^2}}^{y-1} f(x,y)\,dx$$

31. $I = \int_0^2 dx \int_{\sqrt{2x-x^2}}^{2} f(x,y)\,dy$

$$D: \begin{cases} 0 \le x \le 2 \\ \sqrt{2x-x^2} \le y \le 2 \end{cases}$$



$y^2 = 2x - x^2$

$y^2 + (x-1)^2 = 1$

$$D_1^* = \begin{cases} 1 \le y \le 2 \\ 0 \le x \le 2 \end{cases}$$

$(x-1)^2 = 1 - y^2$

$x - 1 = \pm\sqrt{1-y^2}$

$x = 1 \pm \sqrt{1-y^2}$

$$D_2^* = \begin{cases} 0 \le y \le 1 \\ 0 \le x \le 1 - \sqrt{1-y^2} \end{cases}$$

$$D_2^* = \begin{cases} 0 \le y \le 1 \\ 1 + \sqrt{1-y^2} \le x \le 2 \end{cases}$$

$$I = \int_1^2 dy \int_0^2 f(x,y)\,dx + \int_0^1 dy \int_0^{1-\sqrt{1-y^2}} f(x,y)\,dx + \int_0^1 dy \int_{1+\sqrt{1-y^2}}^{2} f(x,y)\,dx$$

32. $\int_0^1 dy \int_{1-\sqrt{1-y^2}}^{2-y} f(x,y)\,dx$

$x = 1 - \sqrt{1-y^2}$

$(x-1) = -\sqrt{1-y^2}$

$k: (x-1)^2 + y^2 = 1$

$k: y^2 = 1 - (x-1)^2$

$y^2 = \cancel{1} - x^2 + 2x - \cancel{1}$

$\boxed{y = \sqrt{2x-x^2}}$

$x = 2 - y$

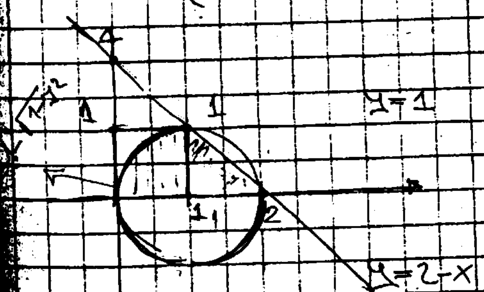
$y = 2 - x$

x	0	2
y	2	0

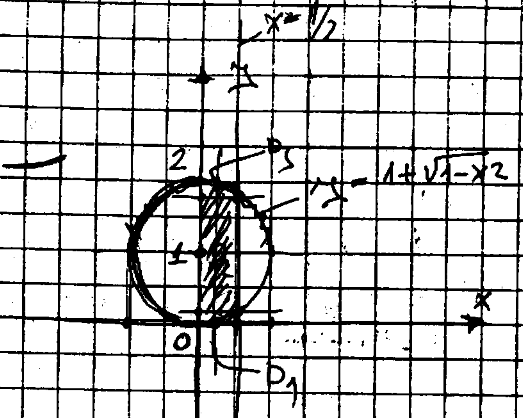
$$D^* = \begin{cases} 0 \le x \le 1 \\ \sqrt{2x-x^2} \le y \le 1 \end{cases}$$

$I = \int_0^1 dx \int_{\sqrt{2x}}$

$$D = \begin{cases} 0 \le y \le 1 \\ 1-\sqrt{1-y^2} \le x \le 2-y \end{cases}$$



133. $\int_0^{1/2} dx \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} f(x,y)\,dy$



$$D = \begin{cases} 0 \le x \le 1/2 \\ 1-\sqrt{1-x^2} \le y \le 1+\sqrt{1-x^2} \end{cases}$$

$y = 1 \pm \sqrt{1-x^2}$

$(y+1)^2 = 1-x^2$

$b:\ (y-1)^2 + x^2 = 1$

$$D_1^* = \begin{cases} 0 \le y \le \frac{2-\sqrt{3}}{2} \\ 0 \le x \le \sqrt{2y-y^2} \end{cases}$$

$$D_2^* = \begin{cases} \frac{2-\sqrt{3}}{2} \le y \le \frac{2+\sqrt{3}}{2} \\ 0 \le x \le 1/2 \end{cases}$$

$$D_3^* = \begin{cases} \frac{2+\sqrt{3}}{2} \le y \le 2 \\ 0 \le x \le \sqrt{2y-y^2} \end{cases}$$

$$I = \int_0^{1-\frac{\sqrt{3}}{2}} dy \int_0^{\sqrt{2y-y^2}} f(x,y)\,dx + \int_{1-\frac{\sqrt{3}}{2}}^{1+\frac{\sqrt{3}}{2}} dy \int_0^{1/2} f(x,y)\,dx + \int_{1+\frac{\sqrt{3}}{2}}^{2} dy \int_0^{\sqrt{2y-y^2}} f(x,y)\,dx$$

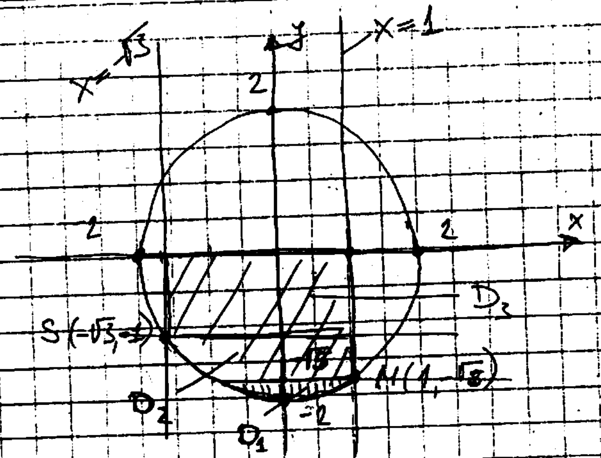
134. $\int_{-\sqrt{3}}^{1} dx \int_{-\sqrt{4-x^2}}^{0} u(x,y)\,dy$

$y = -\sqrt{4-x^2}$

$y^2 = 4-x^2$

$y^2 + x^2 = 4$

$$D = \begin{cases} -\sqrt{3} \le x \le 1 \\ -\sqrt{4-x^2} \le y \le 0 \end{cases}$$



$x=\pm\sqrt{4-y^2}$

$$D_1^* = \begin{cases} -2 \le y \le -\sqrt{3} \\ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \end{cases} \qquad D_2^* = \begin{cases} -\sqrt{3} \le y \le -1 \\ -\sqrt{4-y^2} \le x \le 1 \end{cases}$$

$$D_3^* = \begin{cases} -1 \le y \le 0 \\ -\sqrt{3} \le x \le 1 \end{cases}$$

$$I = \int_{-2}^{-\sqrt{3}} dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx + \int_{-\sqrt{3}}^{-1} dy \int_{-\sqrt{4-y^2}}^{1} f(x,y)\,dx + \int_{-1}^{0} dy \int_{-\sqrt{3}}^{1} f(x,y)\,dx$$

35. $$I = \int_0^1 dy \int_{y^2/2}^{\sqrt{3-y^2}} f(x,y)\,dx$$

$$D = \begin{cases} 0 \le y \le 1 \\ \frac{y^2}{2} \le x \le \sqrt{3-y^2} \end{cases}$$

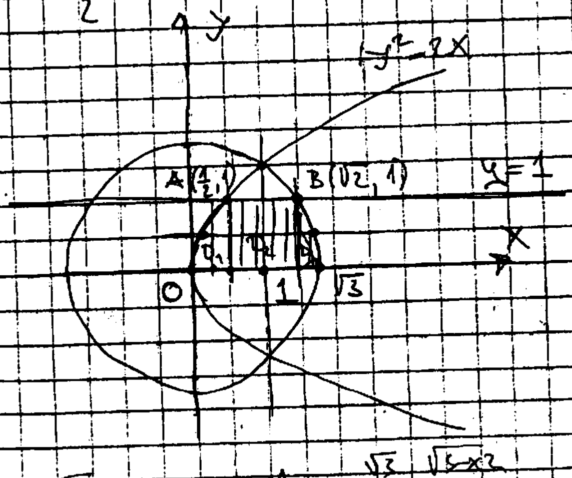
$$\left.\begin{array}{l} x = \frac{y^2}{2} \Rightarrow 2x = y^2 \\ x = \sqrt{3-y^2} \Rightarrow x^2+y^2 = 3 \end{array}\right\} \cap \Rightarrow 2x + x^2 - 3 = 0$$

x^2

$x_{1/2} = \frac{2\pm\sqrt{4+12}}{2}$

$x_{1/2} = \frac{-2\pm 4}{2}$

$x_1 = -3,\ x_2 = 1$



$$D_1^* = \begin{cases} 0 \le x \le \frac{1}{2} \\ 0 \le y \le \sqrt{2x} \end{cases}$$

$$D_2^* = \begin{cases} \frac{1}{2} \le x \le \sqrt{2} \\ 0 \le y \le 1 \end{cases}$$

$$= \int_0^{1/2} dx \int_0^{\sqrt{2x}} f(x,y)\,dy + \int_{1/2}^{\sqrt{2}} dx \int_0^1 dy + \int_{\sqrt{2}}^{\sqrt{3}} dx \int_0^{\sqrt{3-x^2}} dy \qquad D_3^* = \begin{cases} \sqrt{2} \le x \le \sqrt{3} \\ 0 \le y \le \sqrt{3-x^2} \end{cases}$$

$\boxed{36.}$ $\displaystyle\int_0^1 dx \int_{(1-x)^2/2}^{\sqrt{1-x^2}} f(x,y)\,dy$

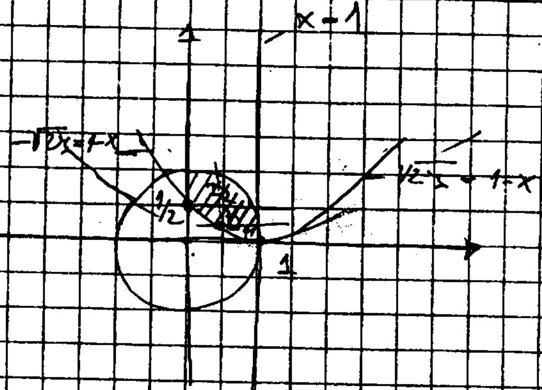
$y = \frac{(1-x)^2}{2}$

$y = \frac{x^2}{2} - x + \frac{1}{2}$

$$D = \begin{cases} 0 \le x \le 1 \\ \frac{(1-x)^2}{2} \le y \le \sqrt{1-x^2} \end{cases}$$

$y = 0$ за $x_{1/2} = 1$

$x^2 + y^2 = 1$



$y = 0$

$(1-x)^2 = 2y$

$1 - x = \pm\sqrt{2y}$

$1 \pm \sqrt{2y} = x$

$x = 1 \pm \sqrt{2y}$

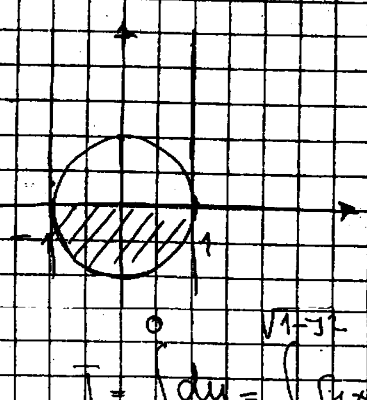
$$D_1^* = \begin{cases} 0 \le y \le 1/2 \\ 1+\sqrt{2y} \le x \le 1 \end{cases}$$

$$D_2^* = \begin{cases} 1/2 \le y \le 1 \\ 0 \le x \le \sqrt{1-y^2} \end{cases}$$

$$\boxed{I} = \int_0^{1/2} dy \int_{1+\sqrt{2y}}^{1} f(x,y)\,dx + \int_{1/2}^{1} dy \int_0^{\sqrt{1-y^2}} f(x,y)\,dx$$

$\boxed{37.}$ $\displaystyle\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{0} f(x,y)\,dy$

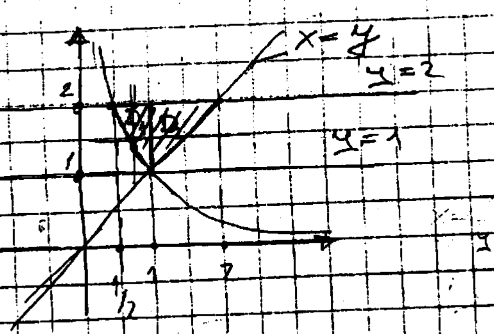
$$D = \begin{cases} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le 0 \end{cases}$$



$$D^* = \begin{cases} -1 \le y \le 0 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{cases}$$

$$I = \int_{-1}^{0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$$

38) $\int_1^2 dy \int_{1/y}^{y} f(x,y)\,dx$



$$D = \begin{cases} 1 \le y \le 2 \\ 1/y \le x \le y \end{cases}$$

$$D_1^* = \begin{cases} 1/2 \le x \le 1 \\ \frac{1}{x} \le y \le 2 \end{cases} \qquad D_2^* = \begin{cases} 1 \le x \le 2 \\ x \le y \le 2 \end{cases}$$

$$I = \int_{1/2}^{1} dx \int_{1/x}^{2} f(x,y)\,dy + \int_{1}^{2} dx \int_{x}^{2} f(x,y)\,dy$$

39) $\int_{-2}^{0} dx \int_{-\sqrt{2-(x^2/2)}}^{0} f(x,y)\,dy$

$y = -\sqrt{2-(x^2/2)}$

$y^2 = 2 - \frac{x^2}{2}$

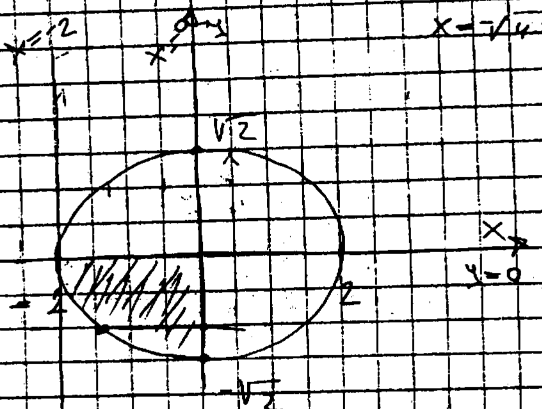
$y^2 = \frac{4-x^2}{2}$ $\qquad 2y^2 + x^2 = 4$

$\frac{x^2}{4} + \frac{y^2}{2} = 1$

$$D = \begin{cases} -2 \le x \le 0 \\ -\sqrt{2-(x^2/2)} \le y \le 0 \end{cases}$$

$x^2 = 4 - 2y^2$

$x = -\sqrt{4-2y^2}$

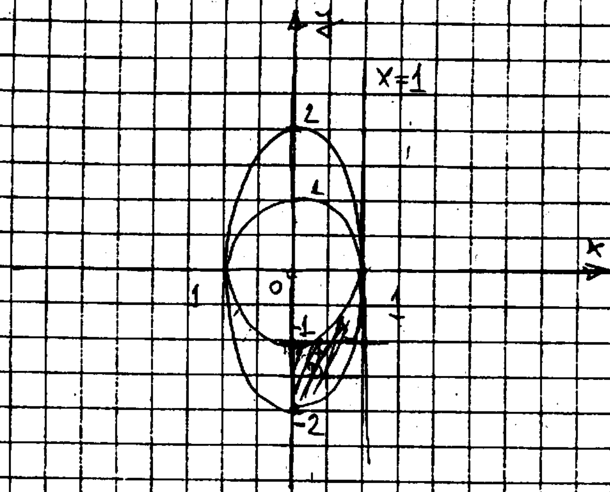


$$D^* = \begin{cases} -\sqrt{2} \le y \le 0 \\ -\sqrt{4-2y^2} \le x \le 0 \end{cases}$$

$$I = \int_{-\sqrt{2}}^{0} dy \int_{-\sqrt{4-2y^2}}^{0} f(x,y)\,dx$$

40. $\int_0^1 dx \int_{-2\sqrt{1-x^2}}^{-\sqrt{1-x^2}} f(x,y)\,dy$

$$D = \begin{cases} 0 \le x \le 1 \\ -2\sqrt{1-x^2} \le y \le -\sqrt{1-x^2} \end{cases}$$



$y = -2\sqrt{1-x^2}$ $\qquad$ $y = -\sqrt{1-x^2}$

$y^2 = 4 - 4x^2$ $\qquad$ $y^2 + x^2 = 1$

$4x^2 + y^2 = 4$

$x = \pm\sqrt{1-y^2}$

$x^2 + \frac{y^2}{4} = 1$

$4x^2 = 4 - y^2$

$$D_1^* = \begin{cases} -1 \le y \le 0 \\ \sqrt{1-y^2} \le x \le \frac{1}{2}\sqrt{4-y^2} \end{cases}$$

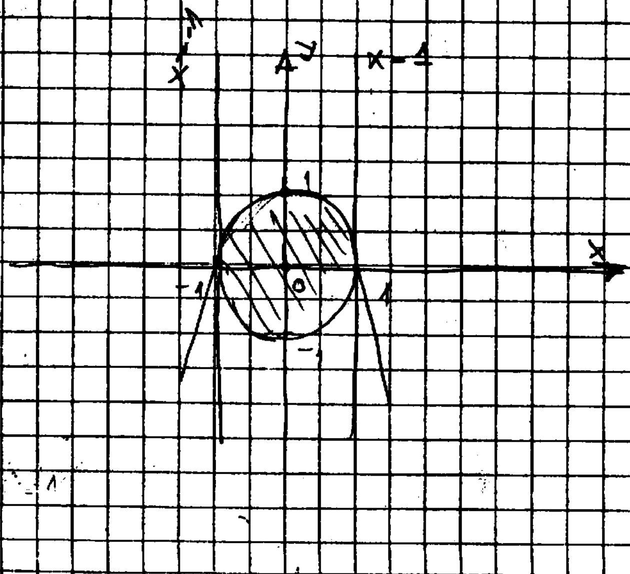
$x^2 = \frac{4-y^2}{4}$

$x = \frac{\pm\sqrt{4-y^2}}{2}$

$$D_2^* = \begin{cases} -2 \le y \le -1 \\ 0 \le x \le \frac{1}{2}\sqrt{4-y^2} \end{cases}$$

$$I = \int_{-1}^{0} dy \int_{\sqrt{1-y^2}}^{\frac{1}{2}\sqrt{4-y^2}} f(x,y)\,dx + \int_{-2}^{-1} dy \int_{0}^{\frac{1}{2}\sqrt{4-y^2}} f(x,y)\,dx$$

41. $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,dy$



$$D = \begin{cases} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le 1-x^2 \end{cases}$$

$y = -\sqrt{1-x^2}$ $\qquad$ $y = 1-x^2$

$y^2 + x^2 = 1$

$$D^* = \begin{cases} -1 \le y \le 1 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{cases}$$

$$I = \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$$

12) $\int_1^2 dx \int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} f(x,y)\,dy$

$y = \sqrt{x^2-1}$

$y^2 = x^2-1$

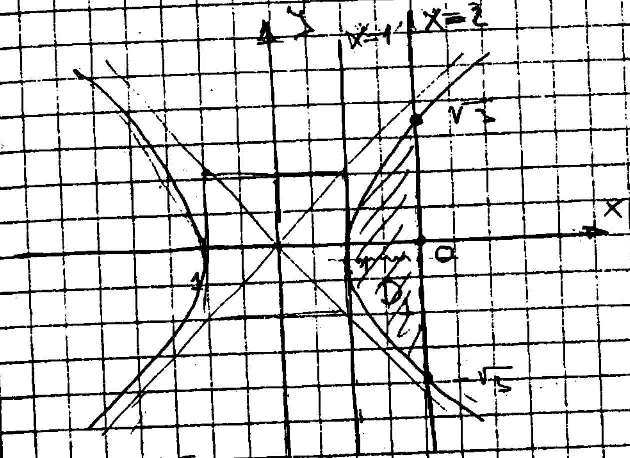
$\frac{x^2}{a} - \frac{y^2}{b} = 1 \to$ хипербола

$D^* = \begin{cases} 1 \le x \le 2 \\ -\sqrt{x^2-1} \le y \le \sqrt{x^2-1} \end{cases}$

$4 - y^2 = 1 - 1$

$3 = y^2$

$y = \pm\sqrt{3}$



$D_1^* = \begin{cases} -\sqrt{3} \le y \le \sqrt{3} \\ \sqrt{1+y^2} \le x \le 2 \end{cases}$

$I = \int_{-\sqrt{3}}^{\sqrt{3}} dy \int_{\sqrt{1+y^2}}^{2} f(x,y)\,dx$

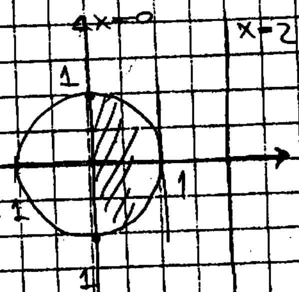
13. $\int_0^2 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy$

$D = \begin{cases} 0 \le x \le 2 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{cases}$

$y = \sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2=1$



$D^* = \begin{cases} -1 \le y \le 1 \\ 0 \le x \le \sqrt{1-y^2} \end{cases}$

$I = \int_{-1}^{1} dy \int_0^{\sqrt{1-y^2}} f(x,y)\,dx$

144) $\int_{-2}^{0} dx \int_{\sqrt{1+x^2}}^{\sqrt{9-x^2}} f(x,y)\,dy$

$$D = \begin{cases} -2 \le x \le 0 \\ \sqrt{1+x^2} \le y \le \sqrt{9-x^2} \end{cases}$$

$y = \sqrt{1+x^2}$ $\quad\quad y^2 -$

$y^2 = 1+x^2$

$y^2 - x^2 = 1$?

145) $\int_{-\sqrt{3}}^{\sqrt{3}} dy \int_{1+\sqrt{1+y^2}}^{3} f(x,y)\,dx$

$$D = \begin{cases} -\sqrt{3} \le y \le \sqrt{3} \\ 1+\sqrt{1+y^2} \le x \le 3 \end{cases}$$

$x = 1 + \sqrt{1+y^2}$

$(x-1)^2 = y^2 + 1$

$\boxed{\underbrace{(x-1)^2 - y^2 = 1}_{\text{хипербола}}}$

$(x-1)^2 - 1 = y^2$

$(x-1+1)(x-1-1) = y^2$

$x(x-2) = y^2$

$x^2 - 2x + \cancel{1} - 1 = y^2$

46. $$\int_{-\pi/2}^{\pi/2} dx \int_{|\sin x|}^{1} f(x,y)\,dy$$

$y = |\sin x|$

$y = \sin x$

$y = -\sin x$

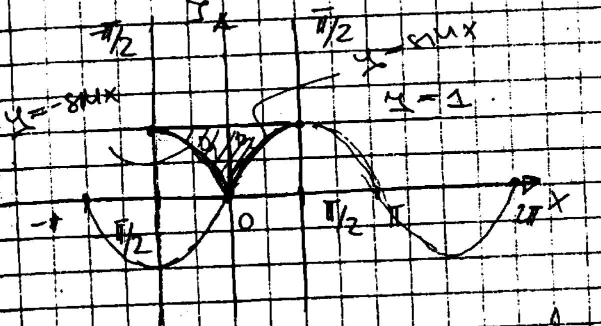
$-y = \sin x$

$x = \arcsin(-y)$

$$D = \begin{cases} -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ |\sin x| \le y \le 1 \end{cases}$$

$$D_1: \begin{cases} 0 \le y \le 1 \\ \arcsin(-y) \le x \le 0 \end{cases}$$

$$D_2: \begin{cases} 0 \le y \le 1 \\ 0 \le x \le \arcsin(y) \end{cases}$$



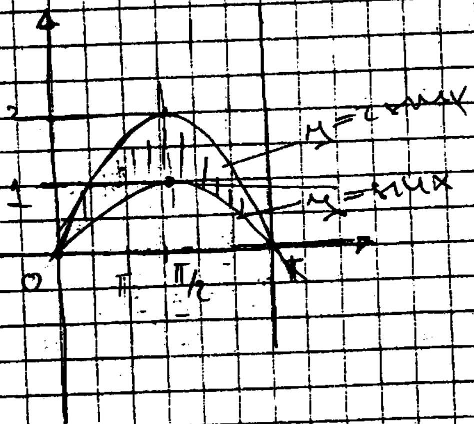
$$I = \int_0^1 dy \int_{\arcsin(-y)}^{0} f(x,y)\,dx + \int_0^1 dy \int_0^{\arcsin(y)} f(x,y)\,dx$$

47. $$\int_0^{\pi} dx \int_{\sin x}^{2\sin x} f(x,y)\,dy$$

$y = \sin x$

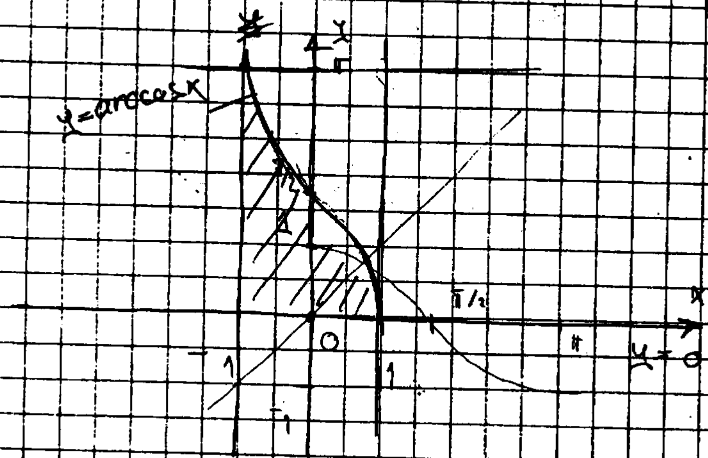
$y = 2\sin x$

$$D = \begin{cases} 0 \le x \le \pi \\ \sin x \le y \le 2\sin x \end{cases}$$



148. $\int_{-1}^{1} dx \int_{0}^{\arccos x} f(x,y)\,dy$

$$D = \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \arccos x \end{cases}$$



$y = \arccos x$

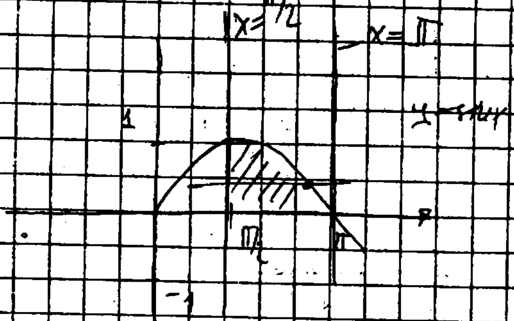
$x = \cos y$

$$D^* = \begin{cases} 0 \le y \le \pi \\ -1 \le x \le \cos y \end{cases}$$

$$I = \int_{0}^{\pi} dy \int_{-1}^{\cos y} f(x,y)\,dx$$

149. $\int_{\pi/2}^{\pi} dx \int_{0}^{\sin x} f(x,y)\,dy$

$$D = \begin{cases} \frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 \le y \le \sin x \end{cases}$$

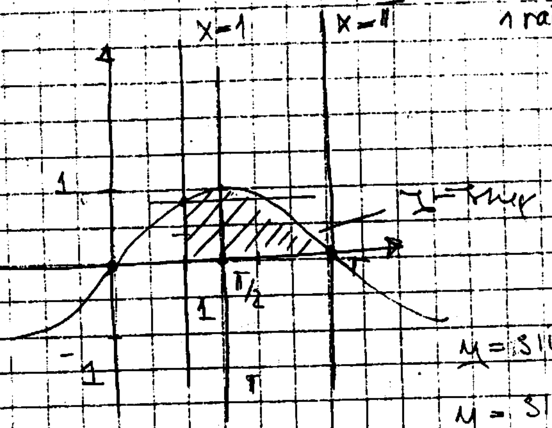


$$D^* = \begin{cases} 0 \le y \le 1 \\ \frac{\pi}{2} \le x \le \arcsin y \end{cases}$$

$$I = \int_{0}^{1} dy \int_{\pi/2}^{\arcsin y} f(x,y)\,dx$$

150. $I=\int_1^{\pi} dx \int_0^{\sin x} f(x,y)\,dy$

$$D=\begin{cases} 1 \le x \le \pi \\ 0 \le y \le \sin x \end{cases}$$



$1 \text{ rad} = \frac{180}{\pi} = 3$

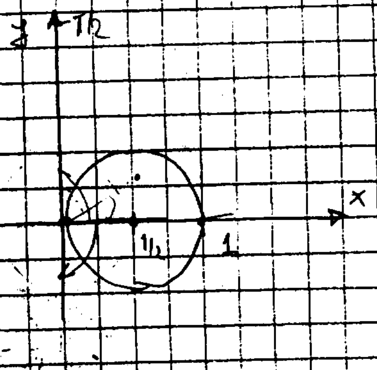
$y = \sin 1$

$y = \sin \frac{180}{\pi} = 57{,}3°$

$1 \text{ rad} = \frac{180°}{\pi} = 57{,}3°$

151. $\int_{-\pi/2}^{\pi/2} d\varphi \int_0^{\cos\varphi} f(\rho,\varphi)\,d\rho$

sin

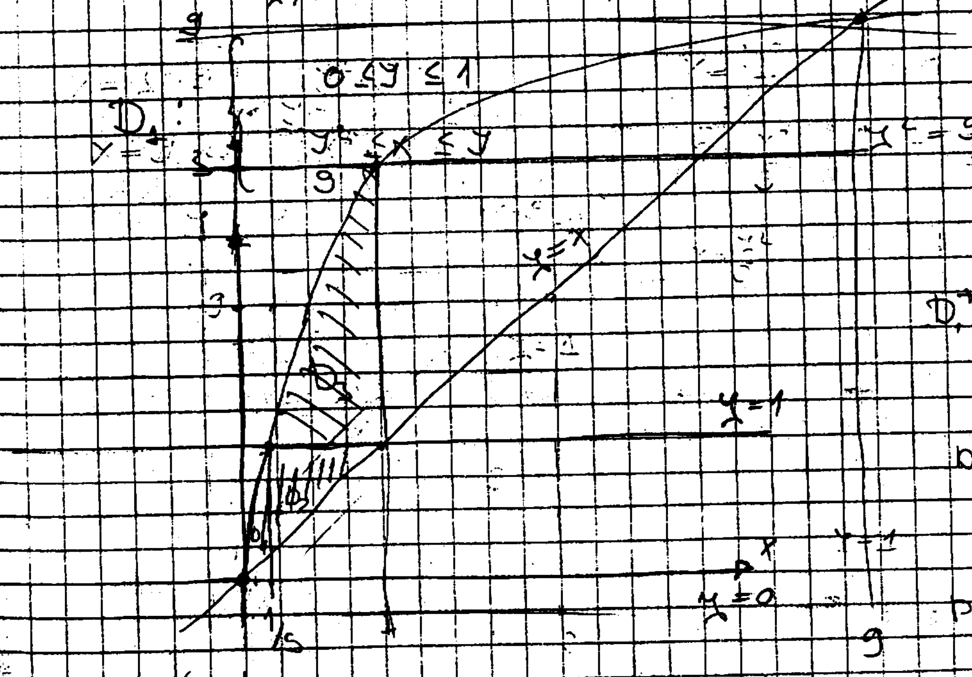


$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

$0 \le \rho \le \cos\varphi$

154. $\int_0^1 dy \int_{y^2/9}^{y} f(x,y)\,dx + \int_1^3 dy \int_{y^2/9}^{1} f(x,y)\,dx$

$$D_1:\begin{cases} 0 \le y \le 1 \\ \frac{y^2}{9} \le x \le y \end{cases}$$



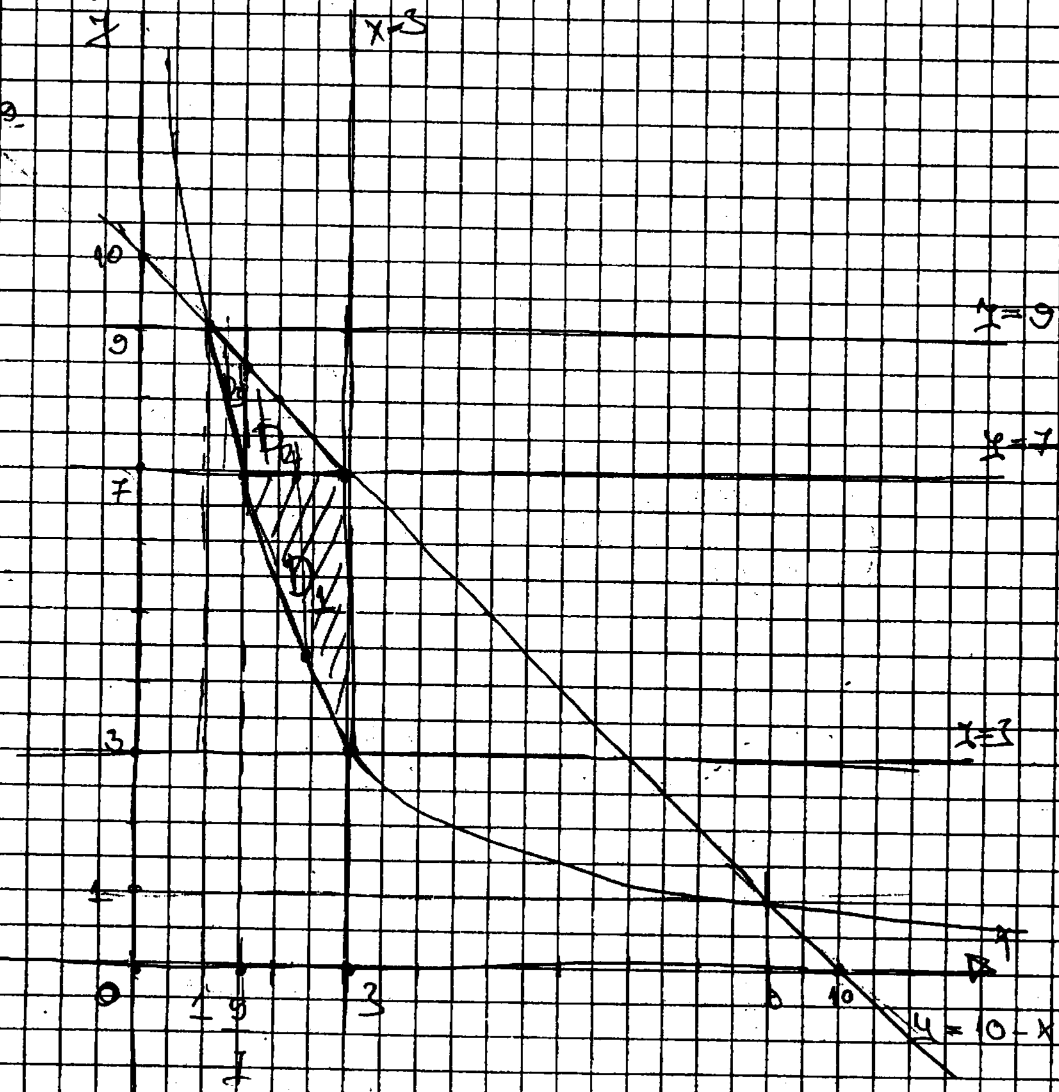
$$D_1^*=\begin{cases} 0 \le x \le 1/9 \\ x \le y \le 3\sqrt{x} \end{cases}$$

$$D_2^*=\begin{cases} 1/9 \le x \le 1 \\ x \le y \le 1 \end{cases}$$

$$D_3^*=\begin{cases} 1/9 \le x \le 1 \\ 1 \le y \le 3\sqrt{x} \end{cases}$$

$\boxed{155.}$ $\int_3^7 dy \int_{9/y}^{3} f(x,y)\,dx + \int_7^9 dy \int_{9/y}^{10-y} f(x,y)\,dx$

$D_1 = \begin{cases} 3 \le y \le 7 \\ \frac{9}{y} \le x \le 3 \end{cases}$ $D_2 = \begin{cases} 7 \le y \le 9 \\ \frac{9}{y} \le x \le 10-y \end{cases}$



$D_1 = \begin{cases} \frac{9}{7} \le x \le 3 \\ \frac{9}{x} \le y \le 7 \end{cases}$ $D_2 = \begin{cases} 1 \le x \le 9/7 \\ \frac{9}{x} \le y \le 10-x \end{cases}$

$D_3 = \begin{cases} \frac{9}{7} \le x \le 3 \\ 7 \le y \le 10-x \end{cases}$

56. $\int_{-2}^{2} dx \int_{0}^{(x+2)/2} f(x,y)\,dy + \int_{2}^{10/3} dx \int_{\sqrt{x^2-4}}^{(x+2)/2} f(x,y)\,dy$

$D_1: \begin{cases} -2 \le x \le 2 \\ 0 \le y \le \frac{x+2}{2} \end{cases}$ $\qquad y = \frac{x}{2} + 1$

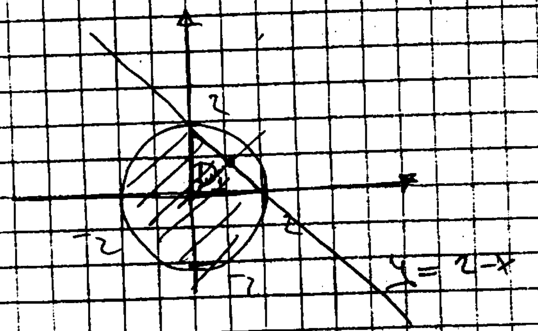
$D_2: \begin{cases} 2 \le x \le 10/3 \\ \sqrt{x^2-4} \le y \le \frac{x+2}{2} \end{cases}$

6.4.1

$D = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 4 \wedge x + y - 2 \le 0 \}$

$y = 2 - x$

x	0	2
y	2	0



$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$\rho^2 = 4$

$\rho = 2$

$\rho\cos\varphi + \rho\sin\varphi - 2 = 0$

$\rho = \frac{2}{\cos\varphi + \sin\varphi}$

$D = D_1 \cup D_2$

$D_1^*: 0 \le \rho \le \frac{2}{\cos\varphi + \sin\varphi}$

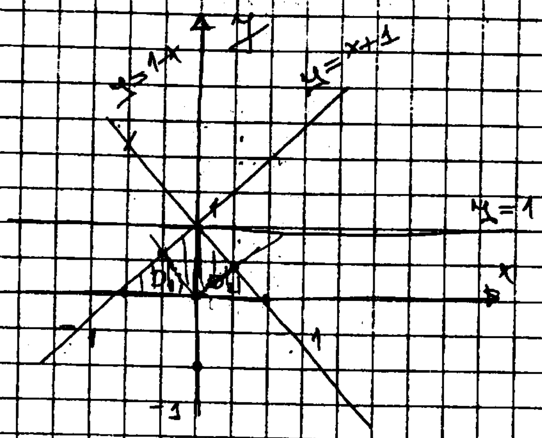
$0 \le \varphi \le \pi/2$

$D_2: 0 \le \rho \le 2$

$\pi/2 \le \varphi \le 2\pi$

$\rho = \frac{2}{\cos\varphi + \sin\varphi}$

19 $D = \{ (x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \ \wedge \ y-1 \le x \le 1-y \}$



$x = y-1 \Rightarrow y = x-1$

x	0	1
y	-1	0

$x = 1-y \Rightarrow x-1 = -y$

$y = 1-x$

x	0	1
y	1	0

$D = D_1 \cup D_2$

$D_1 \Rightarrow \quad \rho\cos\varphi = 1 - \rho\sin\varphi$

$\rho\cos\varphi + \rho\sin\varphi = 1$

$\rho = \dfrac{1}{\cos\varphi + \sin\varphi}$

$D_1: \quad 0 \le \rho \le \dfrac{1}{\cos\varphi + \sin\varphi}$

$0 \le \varphi \le \pi/2$

$\cos\varphi + \sin\varphi = 1$

$\cancel{\cos\varphi = 1 - \sin\varphi}$

$\cancel{\cos\varphi}$

$x = -1$

$\cancel{-1 = \rho\sin\varphi}$

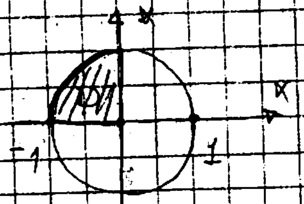
$\varphi =$

$D_2: \quad 0 \le \rho \le \dfrac{1}{\sin\varphi - \cos\varphi}$

$\pi/2 \le \varphi \le \pi$

6.4. A

1) $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1 \ \wedge \ x \le 0 \ \wedge \ y \ge 0 \}$



$$\left.\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right\} k: \boxed{\rho = 1}$$

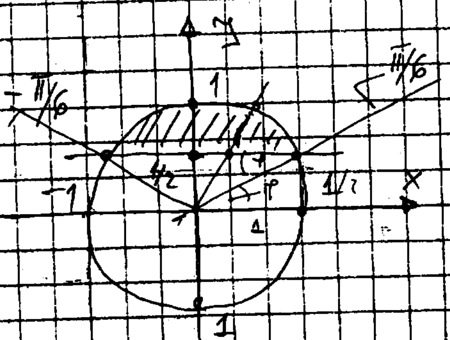
$0 = \rho\cos\varphi \Rightarrow \varphi = \frac{\pi}{2}$

$0 = \rho\sin\varphi \Rightarrow \varphi = \pi$

$0 \le \rho \le 1$

$\pi/2 \le \varphi \le \pi$

2) $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1 \quad y \ge \frac{1}{2} \}$



$\rho^2\sin^2\varphi + \rho^2\cos^2\varphi = 1 \qquad k: \rho = 1$

$y = \frac{1}{2} = \rho\sin\varphi$

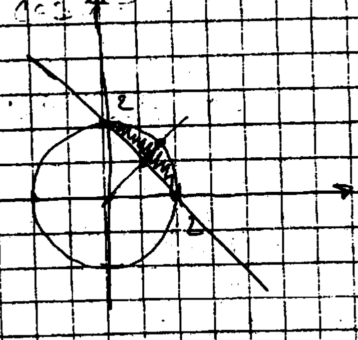
$\rho = \frac{1}{2\sin\varphi}$ ← 2ª curva

$\sin\varphi = \frac{1/2}{1}$

$\frac{1}{2\sin\varphi} \le \rho \le 1$

$-\pi/6 \le \varphi \le \pi/6$

3) $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4 \ \wedge \ x+y-2 \ge 0 \}$



$$\left.\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right\} k: \boxed{\rho = 2}$$

$y = 2 - x$

$\rho\cos\varphi + \rho\sin\varphi - 2 = 0$

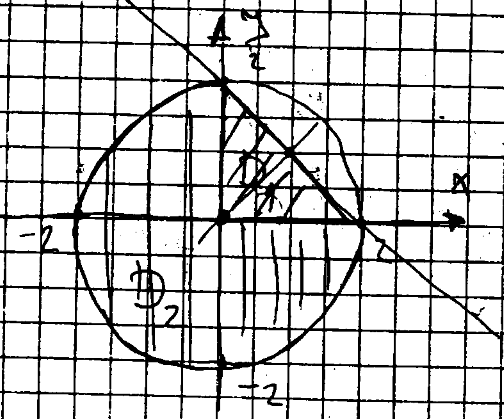
$\rho(\cos\varphi + \sin\varphi) = 2$

2ª: $\rho = \frac{2}{\cos\varphi + \sin\varphi}$

$\frac{2}{\cos\varphi+\sin\varphi} \le \rho \le 2$

$0 \le \varphi \le \pi/2$

[4] $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4 \ \wedge \ x+y-2 \le 0\}$



$k: x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$\underline{k: \rho = 2}$

$y = 2 - x$

$p: \rho = \dfrac{2}{\cos\varphi + \sin\varphi}$

$D_1: \quad 0 \le \rho \le \dfrac{2}{\cos\varphi + \sin\varphi}$

$0 \le \varphi \le \pi/2$

$D_2: \quad 0 \le \rho \le 2$

$\pi/2 \le \varphi \le 2\pi$

[5] $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 9 \ \wedge \ x^2+y^2 \le 6x$

$k_1: \ x^2+y^2 \le 9$

$k_1: \boxed{\rho = 3}$

$k_2: \ x^2 - 6x + y^2 = 0$

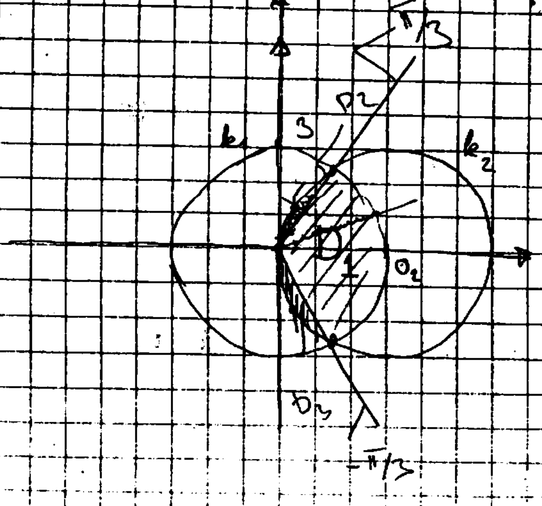
$(x-3)^2 + y^2 = 9$

$\rho^2\cos^2\varphi - 6\rho\cos\varphi + \cancel{9} + \rho^2\sin^2\varphi = \cancel{9}$

$\rho^2 - 6\rho\cos\varphi = 0$

$\rho(\rho - 6\cos\varphi) = 0$

$\boxed{k_2: \rho = 6\cos\varphi}$



$D_1 : \begin{cases} 0 \le \rho \le 3 \\ -\pi/3 \le \varphi \le \pi/3 \end{cases}$

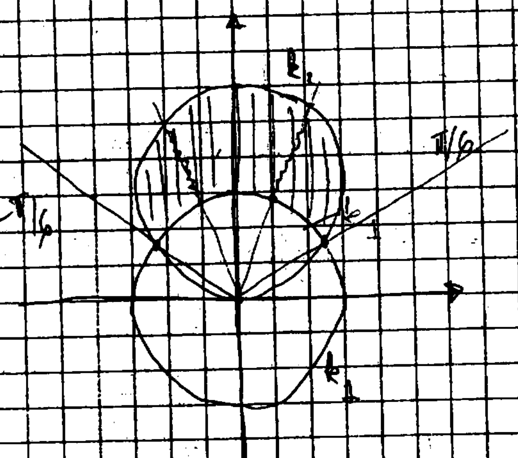
$D_2 : \begin{cases} 0 \le \rho \le 6\cos\varphi \\ \pi/3 \le \varphi \le \pi/2 \end{cases}$

$D_3 : \begin{cases} 0 \le \rho \le 6\cos\varphi \\ -\pi/3 \le \varphi \le -\pi/2 \end{cases}$

$\boxed{6}$ $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \ge 9 \ \wedge \ x^2+y^2 \le 6y\}$

$k_1 : \ x^2+y^2=9 \qquad \boxed{\rho = 3}$

$k_2 : \ x^2+(y-3)^2=9 \qquad \boxed{\rho = 6\sin\varphi}$



$6y - \cancel{y^2} + \cancel{y^2} = 9$

$6y = 9 \quad y = \frac{3}{2}$

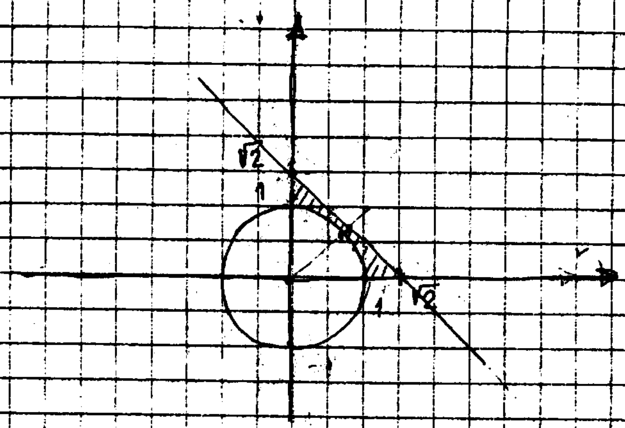
$\frac{3}{2} = 3\sin\varphi \quad \sin\varphi = \frac{1}{2} \quad \varphi = \frac{\pi}{6}$

$3 \le \rho \le 6\sin\varphi$

$-\pi/6 \le \varphi \le \pi/6$

7. $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\geq 1 \wedge x+y\leq\sqrt{2}$

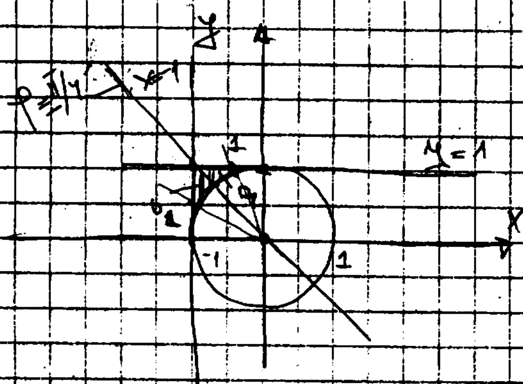
$y=\sqrt{2}-x$



k_1: $\rho=1$ $\qquad 1\leq\rho\leq\frac{\sqrt{2}}{\cos\varphi+\sin\varphi}$

p: $\rho=\frac{\sqrt{2}}{\cos\varphi+\sin\varphi}$ $\qquad 0\leq\varphi\leq\pi/2$

8. $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\geq 1 \wedge -1\leq x\leq 0 \wedge 0\leq y\leq 1\}$



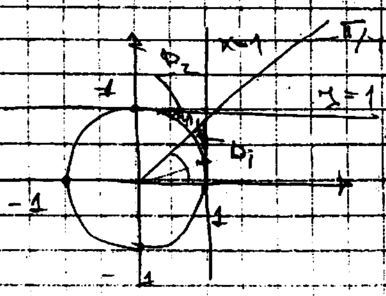
k: $\rho=1$

p_1: $x=-1 \Rightarrow -1=\rho\cos\varphi \Rightarrow \rho=\frac{-1}{\cos\varphi}$

p_2: $y=1 \Rightarrow 1=\rho\sin\varphi \Rightarrow \rho=\frac{1}{\sin\varphi}$

D_2: $\begin{cases} 1\leq\rho\leq\frac{1}{\sin\varphi} \\ -\pi/4\leq\varphi\leq-\pi/2 \end{cases}$ $\qquad D_1$: $\begin{cases} 1\leq\rho\leq-\frac{1}{\cos\varphi} \\ -\pi\leq\varphi-\pi/4 \end{cases}$

9. $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\geq 1 \wedge 0\leq x\leq 1 \wedge 0\leq y\leq 1$

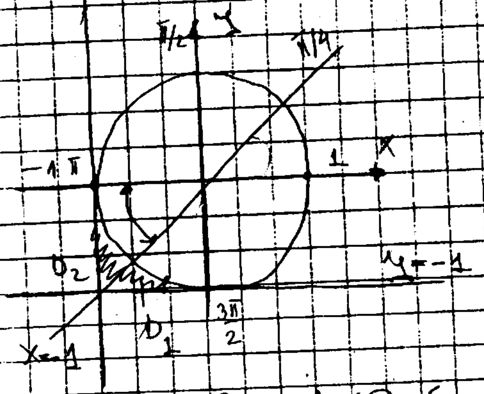


K: $\rho=1$ $\qquad D_1$: $\begin{cases} 1\leq\rho\leq\frac{1}{\cos\varphi} \\ 0\leq\varphi\leq\pi/4 \end{cases}$

$x=1 \Rightarrow \rho=\frac{1}{\cos\varphi}$

$y=1 \Rightarrow \rho=\frac{1}{\sin\varphi}$ $\qquad D_2$: $\begin{cases} 1\leq\rho\leq\frac{1}{\sin\varphi} \\ \pi/4\leq\varphi\leq\pi/2 \end{cases}$

10. $D = \{(x;y) \in \mathbb{R}^2 \mid x^2+y^2 \geq 1 \wedge -1 \leq x \leq 0 \wedge -1 \leq y \leq 0\}$



$k: \rho = 1$

$x = -1: \quad \rho = -\frac{1}{\cos\varphi}$

$y = -1: \quad \rho = -\frac{1}{\sin\varphi}$

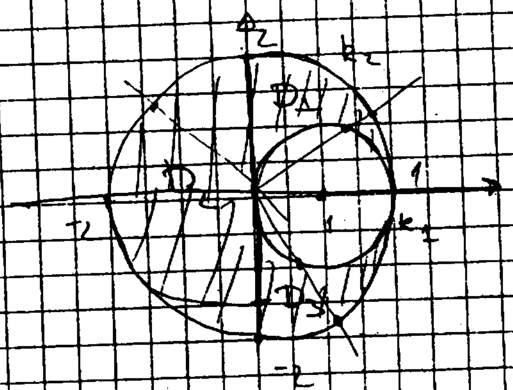
$D_1: \begin{cases} 1 \leq \rho \leq -\frac{1}{\sin\varphi} \\ \leq \varphi \leq \end{cases}$

$D_2: \begin{cases} 1 \leq \rho \leq -\frac{1}{\cos\varphi} \\ \leq \varphi \leq \end{cases}$

11. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4 \wedge x^2+y^2 \geq 2x\}$

$k_1: x^2+y^2 = 4 \quad \boxed{k_1: \rho = 2}$

$k_2: (x-1)^2 + y^2 = 1 \quad \boxed{k_2: \rho = 2\cos\varphi}$



$D_1: \begin{cases} 2\cos\varphi \leq \rho \leq 2 \\ 0 \leq \varphi \quad \pi/2 \end{cases}$

$D_2: \begin{cases} 0 \leq \rho \leq 2 \\ \pi/2 \leq \varphi \quad -\pi/2 \end{cases}$

$D_3: \begin{cases} 2\cos\varphi \leq \rho \leq 2 \\ -\pi/2 \leq \varphi \leq 0 \end{cases}$

12. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq x \wedge x^2+y^2 \leq y\}$

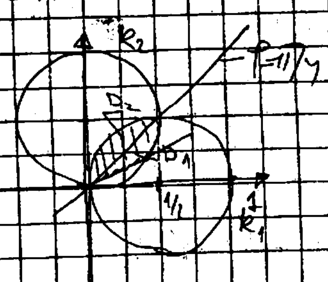
$(x-1/2)^2 + y^2 = \left(\frac{1}{2}\right)^2$

$k_2: x^2 + (y-1/2)^2 = 1/4$

$k_2: \rho = \sin\varphi$

$\rho^2 - \rho\cos\varphi = 0$

$\rho = \cos\varphi$



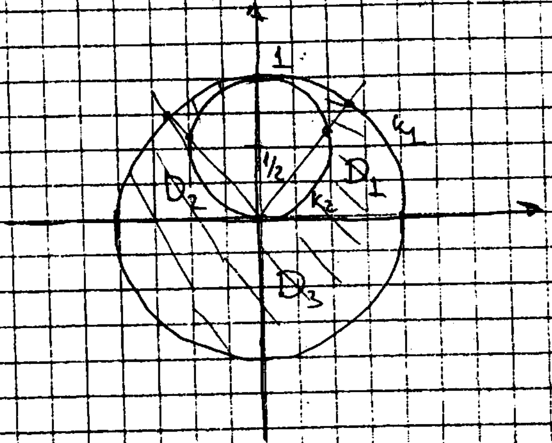
$D_1: \begin{cases} 0 \leq \rho \leq \sin\varphi \\ 0 \leq \varphi \leq \frac{\pi}{4} \end{cases}$

$D_2 = \begin{cases} 0 \leq \rho \leq \cos\varphi \\ \pi/4 \leq \varphi \leq \pi/2 \end{cases}$

13. $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1 \ \wedge\ x^2+y^2\ge y\}$

$k_1:\ x^2+y^2=1 \Rightarrow \boxed{\rho=1}$

$k_2:\ x^2+(y-\frac{1}{2})^2=(\frac{1}{2})^2 \Rightarrow \boxed{\rho=\sin\varphi}$



$D_1:\ \sin\varphi\le\rho\le 1$

$0\le\varphi\le \pi/2$

$D_2:\ \sin\varphi\le\rho\le 1$

$\pi/2\le\varphi\le\pi$

$D_3:\ 0\le\rho\ \ 1$

$\pi\le\varphi\le 2\pi$

14. $D=\{(x,y)\in\mathbb{R}^2 \mid 9x^2+4y^2\le 36 \ \wedge\ x+y\ge 0\}$

$e_1:\ \frac{x^2}{4}+\frac{y^2}{9}=1 \Rightarrow x=a\cos\varphi$

$y=b\sin\varphi$

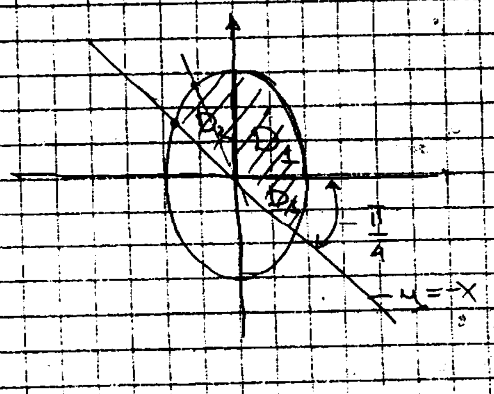
$e:\ \boxed{\rho=1}$

$p:\ \rho\cos\varphi+\rho\sin\varphi=0$

$\Rightarrow \rho\cos\varphi=-\rho\sin\varphi$

$\cos\varphi=-\sin\varphi$

$1=-\operatorname{tg}\varphi \quad -1=\operatorname{tg}\varphi \quad \varphi=$



$0\le$

D_1

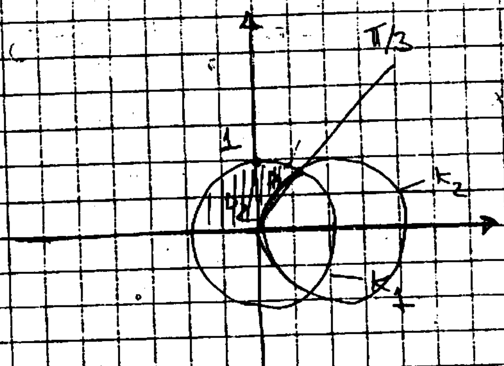
D_2

17. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1 \ \wedge \ x^2+y^2 \ge 2x \ \wedge \ y \ge 0\}$

k_1: $\boxed{\rho = 1}$

k_2: $\boxed{\rho = 2\cos\varphi}$

k_3:



$D_1: \begin{cases} 2\cos\varphi \le \rho \le 1 \\ \pi/3 \le \varphi \le \pi/2 \end{cases}$

$D_2: \begin{cases} 0 \le \rho \le 1 \\ \pi/2 \le \varphi \le \pi \end{cases}$

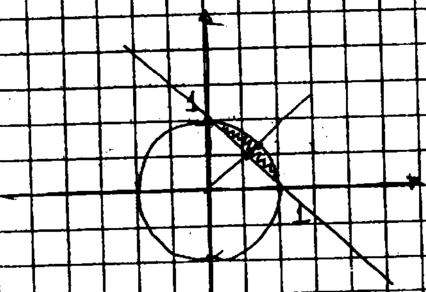
18. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1 \ \wedge \ x+y-1 \ge 0\}$

$y = 1 - x$

k_1: $\rho = 1$

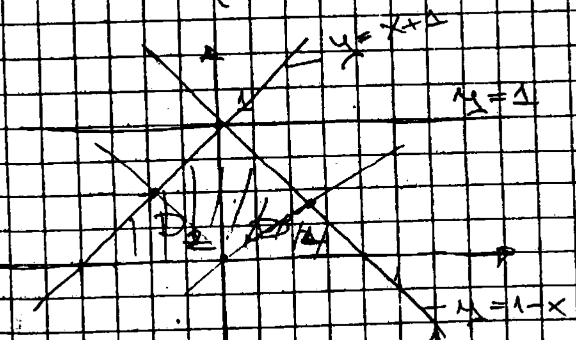
p: $\rho(\cos\varphi + \sin\varphi) = 1$ $\quad \rho = \dfrac{1}{\cos\varphi + \sin\varphi}$

x	0	1
y	1	0



$D = \begin{cases} \dfrac{1}{\cos\varphi+\sin\varphi} \le \rho \le 1 \\ 0 \le \varphi \le \pi/2 \end{cases}$

19. $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \ \wedge \ y-1 \le x \le 1-y\}$



p_2: $y = x + 1$

x	0	-1
y	1	0

p_1: $y = 1 - x$

x	0	1
y	1	0

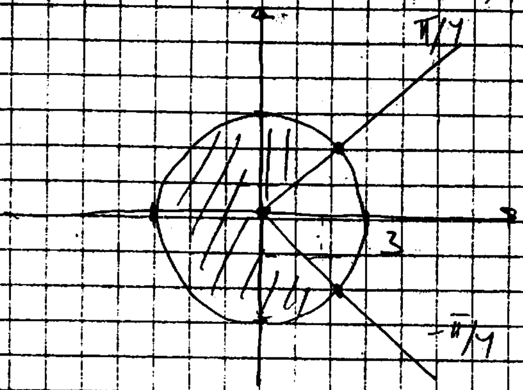
p_1 $\quad \rho = \dfrac{1}{\cos\varphi + \sin\varphi}$

$D_2: \begin{cases} 0 \le \rho \le \dfrac{1}{\sin\varphi - \cos\varphi} \\ \pi/2 \le \varphi \le \pi \end{cases}$ $\quad D_1: \begin{cases} 0 \le \rho \le \dfrac{1}{\cos\varphi + \sin\varphi} \\ 0 \le \varphi \le \pi/2 \end{cases}$

20. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 9 \ \wedge \ x-|y| \ge 0\}$

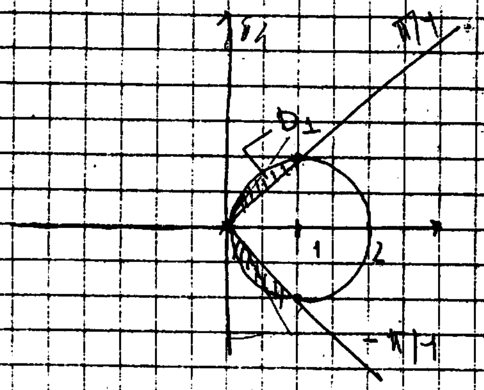
1) $x^2+y^2 = 9 \Rightarrow \rho = 3$

2) $x - |y| = 0 \quad x = |y| \rightarrow$ $\begin{cases} y > 0 & x = y \\ y \le 0 & x = -y \end{cases}$



$D_1 \begin{cases} 0 \le \rho \le 3 \\ -\pi/4 \le \varphi \le \pi/4 \end{cases}$

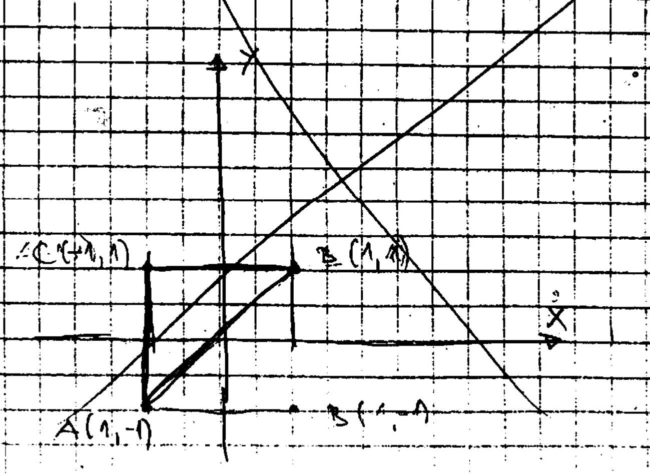
21. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2x \ \wedge \ x-|y| \ge 0\}$



$D_1 : \begin{cases} 0 \le \rho \le 2\cos\varphi \\ \pi/4 \le \varphi \le \pi/2 \end{cases}$

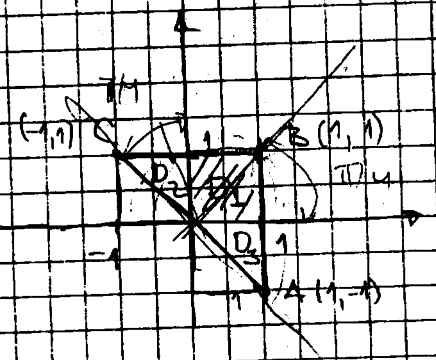
$D_2 \begin{cases} 0 \le \rho \le 2\cos\varphi \\ -\pi/4 \le \varphi \le -\pi/2 \end{cases}$

22. $ABC : A(+1,1) \quad B(1,1) \quad C(-1,1)$



$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

22. ABC, $A(1,-1)$, $B(1,1)$, $C(-1,1)$



$\overline{AB}$: $x = 1$

$\overline{CB}$: $y = 1$

$\overline{CA}$: $y = -x$

$\overline{AB}$: $x = 1$ $\quad 1 = \rho\cos\varphi \Rightarrow \rho = \frac{1}{\cos\varphi}$

$\overline{CB}$: $y = 1$ $\quad 1 = \rho\sin\varphi \Rightarrow \rho = \frac{1}{\sin\varphi}$

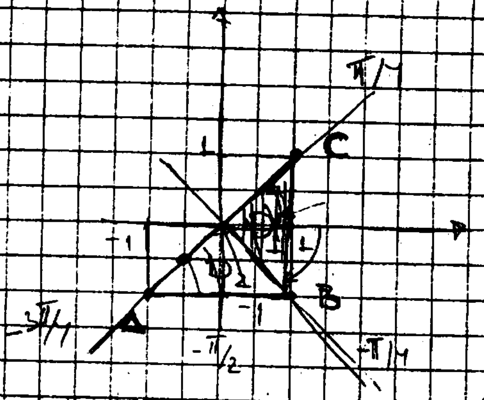
$\overline{CA}$: $-\rho\cos\varphi = \rho\sin\varphi \Rightarrow \operatorname{tg}\varphi = -1$ $\quad \boxed{\varphi = -\pi/4}$

D_1 : $\begin{cases} 0 \le \rho \le 1 \\ 0 \le \varphi \le \pi/2 \end{cases}$

π

D_2 : $\{$

23 $A(-1,-1)$ $B(1,-1)$, $C(1,1)$



$\overline{AB}$: $y=-1$ $\quad -1=\rho\sin\varphi \Rightarrow \rho=\dfrac{-1}{\sin\varphi}$

$\overline{BC}$: $x=1$ $\quad 1=\rho\cos\varphi \Rightarrow \rho=\dfrac{1}{\cos\varphi}$

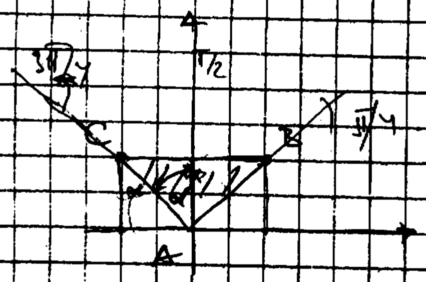
$\overline{AC}$: $y=x$ $\quad \boxed{\varphi=\pi/4}$

D_1: $\begin{cases} 0\le\rho\le\dfrac{1}{\cos\varphi} \\ -\pi/4\le\varphi\le\pi/4 \end{cases}$

D_2: $\begin{cases} 0\le\rho\le\dfrac{1}{\sin\varphi} \\ -\pi/4\le\varphi\le 3\pi/4 \end{cases}$

$5\pi/4\le\varphi\le 7\pi/4$

24 $A(0,0)$, $B(1,1)$, $C(-1,1)$ $\quad \dfrac{\pi}{2}+\dfrac{\pi}{4}=\dfrac{4\pi+\pi}{4}$ $\quad$ ~~$\rho\cos x$~~ $\le\sin x$



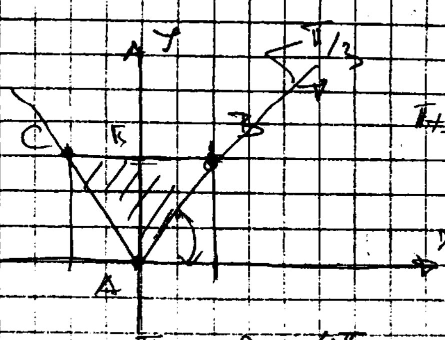
$\overline{AB}\Rightarrow y=x$ $\quad \varphi=\pi/4$

$\overline{AC}\Rightarrow y=-x$

$\overline{CB}\Rightarrow y=1$ $\quad \rho=\dfrac{1}{\cos\varphi}$

$\pi/4\le\varphi\le 3\pi/4$

25 ABC $A(0,0)$ $B(1,\sqrt{3})$, $C(-1,\sqrt{3})$



$\dfrac{\pi+\pi}{3}=\dfrac{4}{3}$

$\overline{AB}-y=\sqrt{3}(x-\cancel{1})$

$y=\sqrt{3}(x-1)$

$\operatorname{tg}\varphi=\sqrt{3}$ $\quad \varphi=\dfrac{\sqrt{3}}{2}$

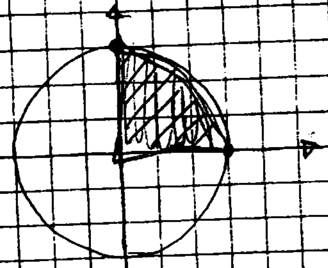
$\varphi=\dfrac{\pi}{2}\Rightarrow \dfrac{1}{2}$

$\pi/3\le\varphi\le\dfrac{4\pi}{3}$

$\boxed{6.4.\ B}$

1) $\iint_D \sqrt{x^2+y^2}\,dx\,dy$ $\quad x^2+y^2=9$

$\boxed{\rho=3}$



$0\le\rho\le 3$

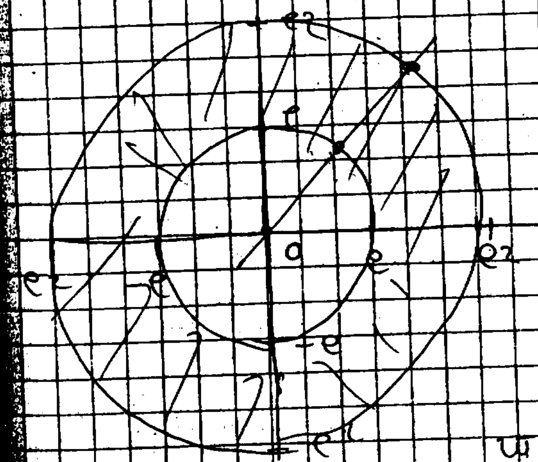
$0\le\varphi\le \pi/2$

$$I=\int_0^{\pi/2}d\varphi\int_0^3\rho\,\rho\,d\rho=\int_0^{\pi/2}d\varphi\,\frac{\rho^2}{2}\Big|_0^3=\int_0^{\pi/2}\frac{9}{2}\,d\varphi=\frac{9}{2}\varphi\Big|_0^{\pi/2}=\frac{9}{2}\cdot\frac{\pi}{2}=\frac{9\pi}{4}$$

2) $\iint_D \ln(x^2+y^2)\,dx\,dy$, $\quad D=\{(x,y)\mid e^2\le x^2+y^2\le e^4\}$

K_1: $x^2+y^2=e^2\Rightarrow \quad \rho^2=e^2 \quad \boxed{\rho=e}\quad K_1$

K_2: $x^2+y^2=e^4 \quad \rho^2=e^4 \quad \boxed{\rho=e^2}$



$e\le\rho\le e^2$

$0\le\varphi\le 2\pi$

$u=\ln\rho$

$du=\frac{d\rho}{\rho}$

$dv=2\,d\rho$

$v=2\rho$

$$\int_0^{2\pi}d\varphi\int_e^{e^2}\ln\rho^2\,d\rho=\int_0^{2\pi}d\varphi\int_e^{e^2}2\ln\rho\,d\rho$$

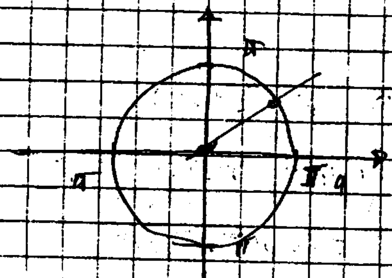
$$=\int_0^{2\pi}d\varphi\left(2\rho\ln\rho\Big|_e^{e^2}-\int_e^{e^2}2\cancel{\rho}\frac{d\rho}{\cancel{\rho}}\right)$$

$$=\int_0^{2\pi}d\varphi\left(2\rho\ln\rho\Big|_e^{e^2}-2\rho\Big|_e^{e^2}\right)$$

$$=\int_0^{2\pi}(2e^2\ln e^2-2e\ln e-2e^2+2e)\,d\varphi=\int_0^{2\pi}(4e^2-\cancel{2e}-2e^2+\cancel{2e})\,d\varphi=\int_0^{2\pi}2e^2\,d\varphi$$

$$=4e^2\pi$$

(3) $\iint\limits_D \left(1-\frac{y^2}{x^2}\right)dx\,dy \qquad D=\{(x,y) \mid x^2+y^2 \le \pi^2\}$



$K_1: \rho^2 = \pi^2 \Rightarrow \boxed{\rho = \pi}$

$0 \le \rho \le \pi$

$0 \le \varphi \le 2\pi$

$\int_0^{2\pi} d\varphi \int_0^{\pi} \left(1-\frac{\sin^2\varphi}{\cos^2\varphi}\right)d\rho$

I_1

$\sin^2\varphi = \frac{1-\cos 2\varphi}{2}$

$\cos^2\varphi - \sin^2\varphi =$

$\frac{1-2\sin^2\varphi}{\cos^2\varphi}$

$I_1 = -\int_0^{\pi} (tg^2\varphi)\,d\varphi$

$-2\,(tg\,\varphi)^2$

$tg\,\varphi = t \qquad d\varphi = \frac{dt}{\sqrt{1+t^2}}$

$tg\,\varphi = t$

$d\varphi = \frac{dt}{\sqrt{1+t^2}}$

$\int \frac{t^2\,dt}{\sqrt{1+t^2}}$

$\int \frac{t^2}{1+t^2}dt = \frac{t^2+1-1}{1+t^2}$

$= \quad 1+t^2 = u \qquad t^2 =$

$2t\,dt = du$

$\int dt - \int \frac{1}{1+t^2}dt$

$t - arctg\,t$

$tg\,\varphi - arctg(tg\,\varphi)$

4.) $\iint\limits_D \frac{dx\,dy}{1+x^2+y^2}$ $\quad y=0 \quad y=\sqrt{1-x^2}$

$1: y=0 \quad \rho\sin\varphi=0 \quad \sin\varphi=\pi$

$\rho\sin\varphi=\sqrt{1-\rho^2\sin^2\varphi}=$

$\rho^2\sin^2\varphi=1-\rho^2\sin^2\varphi$

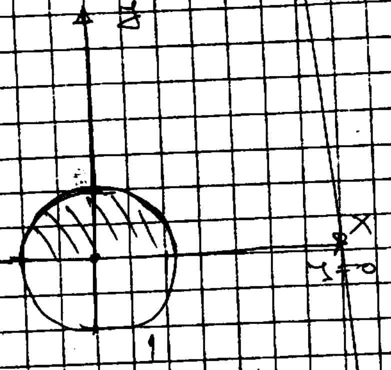
$2\rho^2\sin^2\varphi=1$

$\rho^2\sin^2\varphi=\frac{1}{2}$

$\boxed{\rho=\frac{1}{\sqrt{2}\sin\varphi}}$

$y^2=1-x^2$

$y^2+x^2=1$



$0\le\rho\le\frac{1}{\sqrt{2}\sin\varphi}$

$-\pi\le\varphi\le\pi$

$$\int_{-\pi}^{\pi}d\varphi\int_{0}^{\frac{1}{\sqrt{2}\sin\varphi}}\frac{d\rho}{1+\rho^2}=\int_{-\pi}^{\pi}d\varphi\,(\operatorname{arctg}\rho)\Big|_{\sqrt{2}\sin\varphi}^{1}$$

$\operatorname{arctg}1 \to 1=\operatorname{tg}x \quad x=\pi/4$

$x=\operatorname{arctg}(\sqrt{2}\sin\varphi) \to \sqrt{2}\sin\varphi=\operatorname{tg}x$

$\sqrt{2}\sin\varphi=\operatorname{tg}$

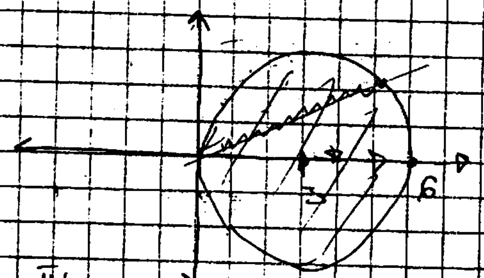
$\sin\varphi=$

5. $\iint_D (x^2+y^2)\,dx\,dy \qquad D=\{(x,y)\mid x^2+y^2\le 6x\}$

$K_1:\ x^2+y^2-6x=0 \qquad (x-3)^2+y^2=9$

$K_1:\ \rho^2\cos^2\varphi+\rho^2\sin^2\varphi-6\rho\cos\varphi=0$

$\rho^2-6\rho\cos\varphi=0 \quad K_1: \boxed{\rho=6\cos\varphi}$



$0\le\rho\le 6\cos\varphi$

$-\pi/2\le\varphi\le\pi/2$

$$\int_{\pi/2}^{\pi/2} d\varphi \int_0^{6\cos\varphi}\rho^2\,d\rho = \int_{-\pi/2}^{\pi/2} d\varphi\ \frac{\rho^3}{3}\Big|_0^{6\cos\varphi}$$

$366-7\,|6|$

$$= 216\int_{-\pi/2}^{\pi/2}\cos^3\varphi\,d\varphi = 216\int_{-\pi/2}^{\pi/2}(1-\sin^2\varphi)\cos\varphi\,d\varphi \qquad \begin{matrix}\sin\varphi=t\\ \cos\end{matrix}$$

$$216\int(1-t^2)\,dt = 216\cdot t-216\cdot\frac{t^3}{3} = 216\cdot\sin\varphi\Big|_{-\pi/2}^{\pi/2} - 72\sin^3\varphi\Big|_{-\pi/2}^{\pi/2}$$

$216\cdot(1-1) \quad -72\,(1 \qquad 432:3=144$

$$= 432\int_0^{\pi/2}\cos^3\varphi\,d\varphi = 432\,t-144\,t^3 = 432\sin\varphi\Big|_0^{\pi/2} - 144\sin^3\varphi\Big|_0^{\pi/2}$$

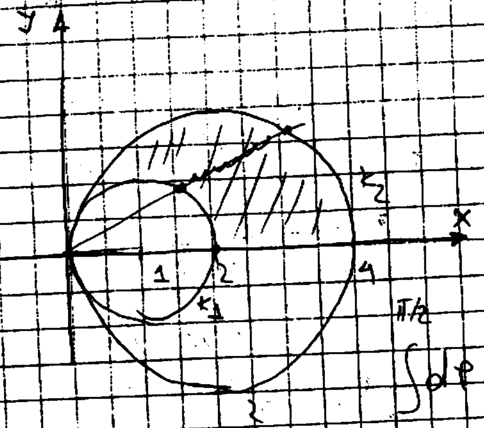
$$=432-144=288$$

6. $\iint_D (x^2+y^2)\,dx\,dy$

$D=\{(x,y)\mid x^2+y^2\ge 2x \wedge x^2+y^2\le 4x \wedge y\ge 0$

$K_1:\ (x-1)^2+y^2=1 \qquad K_1:\ \rho=2\cos\varphi$

$K_2:\ (x-2)^2+y^2=4 \qquad K_2:\ \rho=4\cos\varphi$



$$\int_0^{\pi/2} d\varphi \int_{2\cos\varphi}^{4\cos\varphi} \rho^2 d\rho = \int_0^{\pi/2} d\varphi \, \frac{\rho^3}{3} \Big|_{2\cos\varphi}^{4\cos\varphi}$$

$2\cos\varphi \le \rho \le 4\cos\varphi \quad 0 \le \varphi \le \pi/2$

$$= \int_0^{\pi/2} d\varphi \, \frac{64\cos^3\varphi - 8\cos^3\varphi}{3}$$

$$= \frac{56}{3} \int_0^{\pi/2} \cos^3\varphi = \frac{56}{3} t - \frac{56}{9} t^3 \Big|_0^{\pi/2}$$

$$= \frac{56}{3} \sin\varphi \Big|_0^{\pi/2} - 7 \sin^3\varphi \Big|_0^{\pi/2}$$

$$= \frac{56}{3} - 7$$

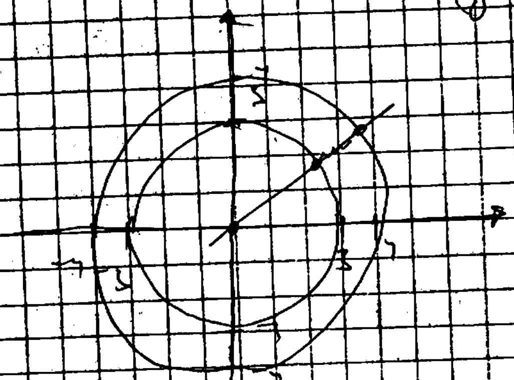
7) $\iint_D (x^2+y^2)\,dx\,dy$

$x^2+y^2 \ge 9 \qquad x^2+y^2 \le 16$

a) $\rho = 3 \qquad (\rho = 4)$

$3 \le \rho \le 4$

$0 \le \varphi \le 2\pi$



$$\int_0^{2\pi} d\varphi \int_3^4 \rho^2 d\rho = \int_0^{2\pi} d\varphi \, \frac{\rho^3}{3} \Big|_3^4 = \int_0^{2\pi} \left(\frac{64-27}{3}\right) d\varphi = \frac{37}{3} \varphi \Big|_0^{2\pi} = \frac{74\pi}{3}$$

$\boxed{8}$ $D = \{(x,y) \mid x^2+y^2 \le 2a^2 \wedge ay \le x^2 \wedge y \ge 0 \wedge x \ge 0$

$k_1: \ x^2+y^2=2a^2$

$\rho^2 = 2a^2$

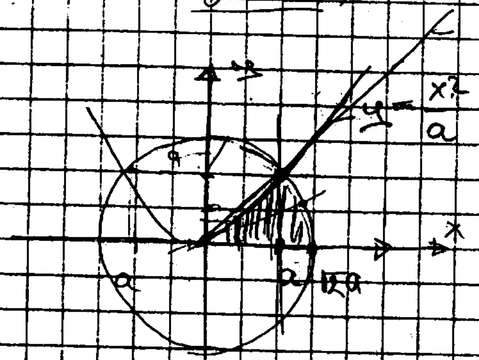
$\boxed{\rho = a\sqrt{2}}$

$k_2: \ ay = x^2$

$a\rho \sin\varphi = \rho^2 \cos^2\varphi$

$a\rho \sin\varphi - \rho^2\cos^2\varphi = 0$

$a\sin\varphi = \rho\cos^2\varphi$



$\rho = \frac{\cos^2\varphi}{a\sin\varphi} = \frac{1-\sin^2\varphi}{a\sin\varphi}$

$\rho = \frac{1}{a\sin\varphi} \quad \frac{1}{a}\sin\varphi$

$ay + y^2 - 2a^2 = 0$

$y^2 + ay - 2a^2 = 0$

$y_{1,2} = \frac{-a \pm \sqrt{a^2+8a^2}}{2} = \frac{-a \pm 3a}{2}$

$y_1 = a \quad a_2 = -2a$

~~$a^2 =$~~

~~$x = \pm\sqrt{ay} = \pm\sqrt{a^2} = a$~~

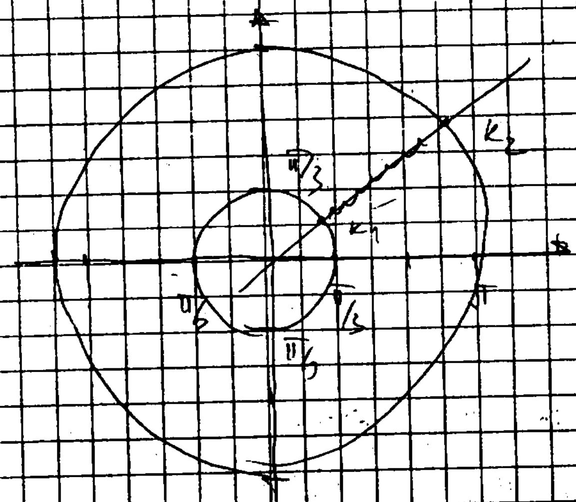
$x = a$

$a = \sin$

9) $\iint\limits_D \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$ $\quad D=\left\{(x,y)\,\middle|\,\frac{\pi^2}{9}\le x^2+y^2\le\pi^2\right\}$

$k_1:\ x^2+y^2=\frac{\pi^2}{9} \Rightarrow \boxed{\rho=\frac{\pi}{3}}$

$k_2:\ x^2+y^2=\pi^2 \Rightarrow \rho=\pi$



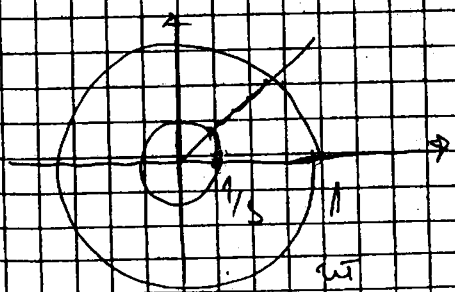
$\frac{\pi}{3}\le\rho\le\pi$

$0\le\varphi\le 2\pi$

$\int\limits_0^{2\pi} d\varphi \int \frac{\sin\rho}{\rho}\,d\rho$

10) $\iint\limits_D \frac{dx\,dy}{\sqrt{x^2+y^2}}$ $\quad D=\left\{(x,y)\,\middle|\,\frac{1}{9}\le x^2+y^2\le 1\right\}$

$k_1:\ x^2+y^2=\frac{1}{9} \Rightarrow \boxed{\rho=\frac{1}{3}}$ $\quad k_2=\boxed{\rho=1}$



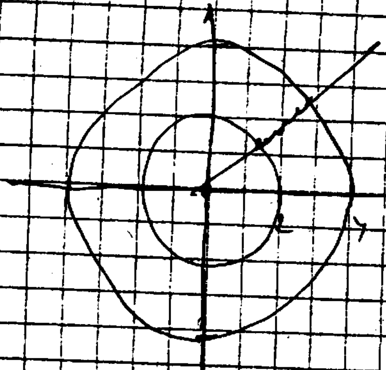
$\int\limits_0^{2\pi} d\varphi \int\limits_{1/3}^{1} \frac{d\rho}{\rho} = \int\limits_0^{2\pi} \ln\rho\Big|_{1/3}^{1}\, d\varphi$

$\int\limits_0^{2\pi}$

$=\int\limits_0^{2\pi} \ln 3^{-1}\, d\varphi = -\ln 3\,\varphi\Big|_0^{2\pi} = -2\pi\ln 3$

$\boxed{11}$ $\iint_D \sqrt{x^2+y^2}\,dx\,dy$ $\quad k_1: x^2+y^2=4 \Rightarrow \rho=2$

$k_2: x^2+y^2=16 \Rightarrow \rho=4$



$2\le\rho\le4$

$0\le\varphi\le2\pi$

$$\int_0^{2\pi}d\varphi\int_2^4\rho\,d\rho=\int_0^{2\pi}d\varphi\cdot\frac{\rho^2}{2}\Big|_2^4=\frac12\int_0^{2\pi}(16-4)\,d\varphi=6\cdot2\pi=12\pi$$

$\boxed{12}$ $\iint_D\sqrt{x^2+y^2-9}\,dx\,dy$ $\quad x^2+y^2=9 \quad \rho=3$

$x^2+y^2=25 \quad \rho=5$

$$\int_0^{2\pi}d\varphi\int_3^5\sqrt{\rho^2-9}\,d\rho \qquad t=\operatorname{arch}\frac{\rho}{3}=$$

$ch^2t-sh^2t=1$

$sh\,t=\sqrt{ch^2t-1}$

$$\int\sqrt{\rho^2-9}\,d\rho=\left[\begin{array}{l}\rho=3\,ch\,t\\ d\rho=3\,sh\,t\,dt\end{array}\right]\Rightarrow ch\,t=\frac{\rho}{3}$$

$sh\,t=\sqrt{\frac{\rho^2}{9}-1}=\frac{\sqrt{\rho^2-9}}{3}$

$$=\int\sqrt{9ch^2t-9}\cdot3\,sh\,t\,dt=\int9\,sh^2t\,dt=9\int\frac{ch2t-1}{2}\,dt$$

$2t=z \quad 2dt=dz$

$$=\frac92\int sh2t\,dt-\frac12\int dt=\frac94\int ch\,z\,dz-\frac92t$$

$$=\frac94sh2t-\frac12t=\frac94\cdot2\,sh\,t\,ch\,t-\frac12t=\frac92\,sh\,t\,ch\,t-\frac92t$$

$$=\frac{\cancel{9}}{2}\,\frac{\sqrt{\rho^2-9}}{\cancel{3}}\cdot\frac{\rho}{\cancel{3}}-\frac92\operatorname{arch}\frac{\rho}{3}$$

$$=\frac{\rho\sqrt{\rho^2-9}}{2}-\frac92\ln\left|\frac{\rho}{3}+\frac{\sqrt{\rho^2-9}}{3}\right|$$

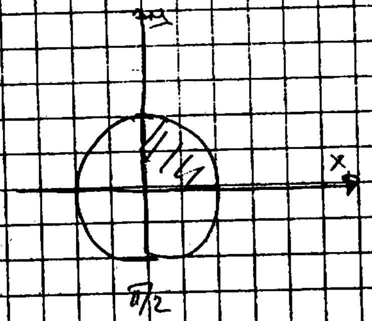
$$\int_0^{2\pi} d\varphi \left(\frac{\rho^2}{2}\sqrt{\rho^2-9} - \frac{9}{2}\ln\left|\frac{\rho+\sqrt{\rho^2-9}}{3}\right| \right)\Bigg|_3^5$$

$$\int_0^{2\pi} \left(\frac{5}{2}\sqrt{16} - \frac{9}{2}\ln\left|\frac{5+\sqrt{16}}{3}\right| \right) d\varphi = \int_0^{2\pi} \left(10 - \frac{9}{2}\ln 3 \right) d\varphi$$

$$= \left(10 - \frac{9}{2}\ln 3\right)\cdot 2\pi = 20\pi - 9\pi\ln 3$$

13) $\iint_D \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$ $\quad D = \{ (x,y) \mid x^2+y^2 \le 1 \quad x \ge 0 \quad y \ge 0 \}$

$\boxed{\rho = 1}$



$0 \le \rho \le 1$

$0 \le \varphi \le \pi/2$

$\arcsin 1 = x$

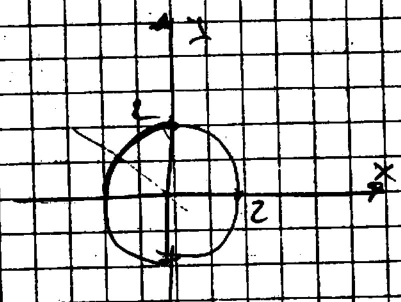
$\sin x = 1 \Rightarrow \pi/2$

$$\int_0^{\pi/2} d\varphi \int_0^1 \frac{d\rho}{\sqrt{1-\rho^2}} = \int_0^{\pi/2} d\varphi \,(\arcsin\rho)\Big|_0^1 =$$

$$= \int_0^{\pi/2} (\arcsin 1 - \arcsin 0)\, d\varphi = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$$

14) $\iint_D x\,dx\,dy$, $\quad D = \{ (x,y) \mid x^2+y^2=4 \quad x \le 0 \wedge y \ge 0)$

$\rho = 2$



$x = \rho\cos\varphi$

$0 = 2\cos\varphi$

$\varphi = +\pi$

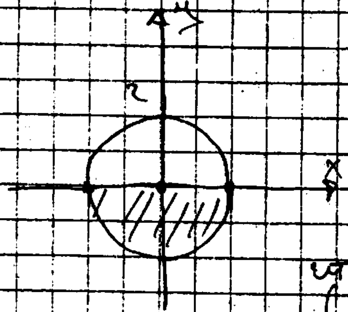
$+\pi \le \varphi \le \pi/2$

$2 \le \rho \le 2$

$$\int_{\pi}^{\pi/2} d\varphi \int_2^2 \rho\cos\varphi \, d\rho =$$

16-329
7-225

15. $\iint_D xy\,dx\,dy$, $D=\{(x,y) \mid x^2+y^2\le 4 \ \wedge \ y\le 0\}$



$0\le \rho \le 2$

$\pi \le \varphi \le 2\pi$

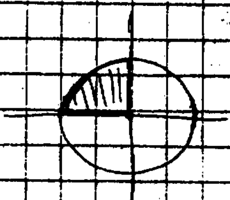
$$\int_\pi^{2\pi} d\varphi \int_0^2 \rho^2 \sin\varphi\cos\varphi\, d\rho$$

$$\int_\pi^{2\pi} \sin\varphi\cos\varphi \, \frac{\rho^3}{3}\Big|_0^2 = \int_\pi^{2\pi} 8\sin\varphi\cos\varphi\, d\varphi$$

$$= 8\int_\pi^{2\pi} t\,dt = 4t^2\Big|_\pi^{2\pi} = 4\sin^2\varphi\Big|_\pi^{2\pi} = 0$$

$\sin\varphi = t$
$\cos\varphi\, d\varphi = dt$

17. $\iint_D x\,d$



$0\le \rho \le 2$

$\pi/2 \le \varphi \le \pi$

$$\iint_D x\,dx\,dy = \int_{\pi/2}^{\pi} d\varphi \int_0^2 \rho\cos\varphi\, d\rho$$

$$= \int_{\pi/2}^{\pi} d\varphi \cos\varphi \cdot \frac{\rho^2}{2}\Big|_0^2 = 2\int_{\pi/2}^{\pi} \cos\varphi\, d\varphi = 2\sin\varphi\Big|_{\pi/2}^{\pi}$$

$$= -2$$